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Springer

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ISSN 1864-5879 ISSN 1864-5887 (electronic)
Theoretical and Mathematical Physics
ISBN 978-3-319-74490-2 ISBN 978-3-319-74491-9 (eBook)
https://doi.org/10.1007/978-3-319-74491-9

Library of Congress Control Number: 2018930138

Printed on acid-free paper

This Springer imprint is published by the registered company Springer International Publishing AG part of Springer Nature
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

This book is dedicated to my beloved daughter Laura, to my darling wife Olga and to my young son Vladimir. My deep feelings of love are hereby transmitted to the three of them, with all the hopes and the worries of an old father who stubbornly believes that culture and science are just one thing and encode, for mankind, the unique escape route from global disasters.

Preface

This book forms a twin pair with another book [1] by the same author, which is of a different, historically oriented character, while the character of the present volume is thoroughly mathematical. These two pieces of work constitute a twin pair since, notwithstanding their different profile and contents, they arise from the same vision and pursue complementary goals.

The vision, extensively discussed in [1], consists of the following main conceptual assessments:

1. Our current understanding of the Fundamental Laws of Nature is based on a coherent, yet provisional, set of five meta-theoretical principles, listed by me as (A)–(E) and dubbed the current *episteme*. This episteme is of genuine geometrical nature and can be viewed as the current evolutionary state of Einstein's ideas concerning the geometrization of physics.
2. *Geometry* and *Symmetry* are inextricably entangled, and their current conception is the result of a long process of abstraction, traced back in [1], which was historically determined and makes sense only within the *Analytic System of Thought* of Western Civilization, started by the ancient Greeks.
3. The evolution of *Geometry and Symmetry Theory* in the last forty years has been deeply and very much constructively influenced by *Supersymmetry/Supergravity* and the allied constructions of *Strings and Branes*.
4. Further advances in Theoretical Physics cannot be based simply on the Galilean Method of interrogating first Nature and then formulating a testable theory that explains the observed phenomena. As stated in [1], one ought to interrogate also *Human Thought*, by this meaning frontier-line mathematics concerned with geometry and symmetry in order to find there the threads of so far unobserved correspondences, reinterpretations, and renewed conceptions.

The complementary pursued goals are:

(a) In the case of book [1]

- the historical and conceptual analysis of the process mentioned in point (2) of the above list which led to the current episteme.

- the philosophical argumentation, on historical basis, of the assessment made in point (4) of the above list.

(b) In the case of the present book, the mathematical full-fledged illustration of the main developments in geometry and symmetry theory that occurred under the fertilizing influence of *Supersymmetry/Supergravity* and that would be inconceivable without the latter.

In view of this, it is reasonable to quote from the ample discussion presented in [1] the summary of the current *episteme* as I understand it. There I say what follows.

The Episteme

As a theoretical physicist, I consider myself very fortunate to have witnessed, in my own lifetime, the following series of experimental discoveries:

1. The detection of the $W^{\pm}$ and Z particles, definitely confirming that fundamental non-gravitational interactions can be described by gauge theories.
2. The detection of the Brout Englert Higgs boson, definitely confirming that gauge theories can be spontaneously broken by scalar fields falling into non-symmetric extrema of some potential.
3. The direct detection of gravitational waves emitted in the coalescence of two compact stars (black holes or neutron stars) which not only confirms the general structure of General Relativity, but directly tests the dynamics encoded in Einstein Equations, namely in a set of purely geometrical differential equations.

Trying to summarize the implications for the *episteme* of the last thirty-three years of experimental physics, we can say the following.

Leaving apart the issue of *quantization* that we can generically identify with the *functional path integral over classical configurations*, we have, within our Western Analytic System of Thought, a rather simple and universal scheme of interpretation of the Fundamental Interactions and of the Fundamental Constituents of Matter based on the following few principles:

(A) The categorical reference frame is provided by Field Theory defined by some action $\mathscr{A} = \int_{\mathscr{M}} \mathscr{L}(\Phi, \partial\Phi)$ where $\mathscr{L}(\Phi, \partial\Phi)$ denotes some Lagrangian depending on a set of fields $\Phi(x)$.

(B) All fundamental interactions are described by *connections* $\mathbf{A}$ on *principal fiber bundles* $P(\mathrm{G}, \mathscr{M})$ where G is a Lie group and the base manifold $\mathscr{M}$ is some space-time in $d = 4$ or in higher dimensions.

(C) All the fields Φ describing fundamental constituents are *sections* of *vector bundles* $B(\mathrm{G}, V, \mathscr{M})$, associated with the principal one $P(\mathrm{G}, \mathscr{M})$ and determined by the choice of suitable *linear representations* $D(\mathrm{G}) : V \rightarrow V$ of the structural group G.

(D) The spin-zero particles described by scalar fields ϕ^I have the additional feature of admitting nonlinear interactions encoded in a scalar potential $\mathscr{V}(\phi)$ for whose choice general principles, supported by experimental confirmation, have not yet been determined.

(E) Gravitational interactions are special among the others and universal since they deal with the *tangent bundle* $T\mathscr{M} \to \mathscr{M}$ to space-time. The relevant connection is in this case the Levi-Civita connection (or some of its generalization with torsion) which is determined by a metric g on $\mathscr{M}$.

A quick look at the list of principles (A)–(E) immediately reveals that, notwithstanding their simplicity and unifying power, they can be only provisional. There are still too many ad hoc choices which strongly demand some deeper unifying principle able to predict them from above. Most prominent among these choices are those of the structural group G, of the representations $D(\mathrm{G})$ and of the potential $\mathscr{V}(\phi)$, the latter choice including also, in some extended sense, the determination of quark and lepton masses. What I have described in the above way is described in the physical literature of the last forty years as the problem of *grand unification* or of *super unification*.

Supersymmetry-Inspired Trends in Geometry and Group Theory

In the same forty years, an enormously extended set of developments have taken place in the quest for unification, starting from the new idea of *Supersymmetry* which, as the word reveals, is an extension of the notion of *Symmetry*, meaning by that Lie algebras. The reason why Supersymmetry, which leads to the fields of *Supergravity*, *Superstrings,* and *Brane Physics,* entrains so many structural and ramified implications is because it tackles with one of the most fundamental and, in my opinion, not yet fully penetrated, principles of physics, namely the distinction among fermion and bosons, intertwined, by means of the spin–statistics theorem with Lie algebra theory, the distinction between two groups of representations, the vector and the spinor ones, being a distinctive property of the $\mathfrak{so}(n)$ Lie algebras, unexisting for the others.

The largest part of the developments mentioned above, related with Supergravity/Superstrings, have a distinctive geometrical/algebraic basis. Entire chapters of algebraic geometry and of algebraic topology have been integrated by these developments into the fabrics of theoretical physics, while some new geometries have been introduced into the fabrics of mathematics. Furthermore, the very way to analyze and interpret mathematical structures is sometimes redirected by the influence of Supergravity/Superstrings. Two or three examples suffice to illustrate what I mean. Exceptional Lie algebras that, up to the mid-1960s were considered by the majority of physicists like mathematical curiosities, have been promoted to the role of primary actors on the stage of the *Superworld*. Special Kähler geometries, never defined by pure mathematicians have by now entered, with full rights, the mathematical club, revealing their relation with other geometries, already introduced by mathematicians, like HyperKähler geometry and quaternionic Kähler geometry. The notions of momentum map, Kähler, and HyperKähler quotients find a deep interpretation in the context of supersymmetric field theories and connect with some of the most brilliant mathematical achievements of the last few decades like the Kronheimer construction of ALE manifolds.

The Topics of the Present Volume and Its Mission

Relying on the above arguments and explanations, I can now more appropriately restate the topic of the present book, which is *the scope of Group Theory, of the Differential Geometry of Coset Manifolds and of various issues in Special Geometries* as they have been promoted and assessed under the influence of current research in Supergravity.

In line with above the statements, it goes without saying that the education of present time physicists, in particular theoretical, but not only, should include, from a very early stage of their student career, a ground course in the basic *Mathematics of Symmetry*, namely in group theory, discrete and Lie groups being equally essential, and in the fundaments of differential geometry. Such course should be mathematically precise, yet more focused on the fundamental mathematical ideas than on the task of mathematical rigorous proofs. Furthermore, it should provide explicit constructions and train the student in the art of explicit calculations, especially those implemented on computers. To such a task is devoted the textbook [2] which was recently published.

Repeating my words in a slightly different form, I think that what is currently practiced in the whole world as Fundamental Physics or Mathematics is based on the Greek view of the *episteme* and it is meaningful only inside the Analytic System of Thought founded by the ancient Greeks. To recuperate a full conscience of this fact is mandatory in order to continue on the difficult but exciting path we are confronted with.

The twin pair of which this book is a member, together with the more introductory textbook [2], is viewed by the author as his limited, humble contribution to the promotion of a new season of more scholarly teaching of *physical mathematics*.

Spes, ultima dea.

Turin, Italy
November 2017

Pietro Giuseppe Fré

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1. P.G. Fré, *A Conceptual History of Symmetry from Plato to the Superworld* (Springer, Berlin, 2018)
2. P.G. Fré, A.M. FedotoBv, *Groups and Manifolds, Lectures for Physicists with examples in Mathematica* (De Gruyter, Berlin, 2018)

Acknowledgements

With great pleasure, I would like to thank my collaborators and dear friends Pietro Antonio Grassi, Ugo Bruzzo, Alexander Sorin, and Mario Trigiante for several discussions, continuous encouragement, and many precious consultations. It is also my pleasure to acknowledge important discussions with my great distinguished friend Sergio Ferrara, in relation with the several common papers that are reflected in the texture of the present book quite ubiquitously. Similarly, I express my gratitude to the Editors of Springer-Verlag, in particular to Maria Bellantone for their continuous assistance, constructive criticism, and suggestions.

My thoughts, while finishing the writing of this long essay that occurred in Moscow, were frequently directed to my late parents, whom I miss very much and I will never forget. To them, I also express my gratitude for all what they taught me in their life, in particular to my father who, with his own example, introduced me, since my childhood, to the great satisfaction and deep suffering of writing books.

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