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Andreas M. Hinz • Sandi Klavžar • Ciril Petr

The Tower of Hanoi – Myths and Maths

Second Edition

Foreword by Ian Stewart

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Andreas M. Hinz
Faculty of Mathematics,
Computer Science and Statistics,
LMU München
Munich, Germany

Sandi Klavžar
Faculty of Mathematics and Physics
University of Ljubljana
Ljubljana, Slovenia

Ciril Petr
Faculty of Natural Sciences and Mathematics
University of Maribor
Maribor, Slovenia

ISBN 978-3-319-73778-2 ISBN 978-3-319-73779-9 (eBook)
https://doi.org/10.1007/978-3-319-73779-9

Library of Congress Control Number: 2018935596

Printed on acid-free paper

This book is published under the imprint Birkhäuser, www.birkhauser-science.com by the registered company Springer International Publishing AG part of Springer Nature.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Foreword

by Ian Stewart

I know when I first came across the Tower of Hanoi because I still have a copy of the book that I found it in: *Riddles in Mathematics* by Eugene P. Northrop, first published in 1944. My copy, bought in 1960 when I was fourteen years old, was a Penguin reprint. I devoured the book, and copied the ideas that especially intrigued me into a notebook, alongside other mathematical oddities. About a hundred pages further into Northrop's book I found another mathematical oddity: Wacław Sierpiński's example of a curve that crosses itself at every point. That, too, went into the notebook.

It took nearly thirty years for me to become aware that these two curious structures are intimately related, and another year to discover that several others had already spotted the connection. At the time, I was writing the monthly column on mathematical recreations for *Scientific American*, following in the footsteps of the inimitable Martin Gardner. In fact, I was the fourth person to write the column. Gardner had featured the Tower of Hanoi, of course; for instance, it appears in his book *Mathematical Puzzles and Diversions*.

Seeking a topic for the column, I decided to revisit an old favourite, and started rethinking what I knew about the Tower of Hanoi. By then I was aware that the mathematical essence of many puzzles of that general kind—rearranging objects according to fixed rules—can often be understood using the state diagram. This is a network whose nodes represent possible states of the puzzle and whose edges correspond to permissible moves. I wondered what the state diagram of the Tower of Hanoi looked like. I probably should have thought about the structure of the puzzle, which is recursive. To solve it, forget the bottom disc, move the remaining ones to an empty peg (the same puzzle with one disc fewer), move the bottom disc, and put the rest back on top. So the solution for, say, five discs reduces to that for four, which in turn reduces to that for three, then two, then one, then zero. But with no discs at all, the puzzle is trivial.

Instead of thinking, I wrote down all possible states for the Tower of Hanoi with three discs, listed the legal moves, and drew the diagram. It was a bit messy, but after some rearrangement it suddenly took on an elegant shape. In fact, it looked remarkably like one of the stages in the construction of Sierpiński's curve. This couldn't possibly be coincidence, and once I'd noticed this remarkable resemblance, it was then straightforward to work out where it came from: the recursive structure of the puzzle.

Several other people had already noticed this fact independently. But shortly after my rediscovery I was in Kyoto at the International Congress of Mathematicians. Andreas Hinz introduced himself and told me that he had used the connection with the Tower of Hanoi to calculate the average distance between any two points of Sierpiński's curve. It is precisely 466/885 of the diameter. This is an extraordinary result—a rational number, but a fairly complicated one, and far from obvious.

This wonderful calculation is just one of the innumerable treasures in this fascinating book. It starts with the best account I have ever read of the history of the puzzle and its intriguing relatives. It investigates the mathematics of the puzzle and discusses a number of variations on the Tower of Hanoi theme. This new edition has been updated with the latest discoveries, including Thierry Bousch's impressive proof that the conjectured minimum number of moves to solve the four-tower version is correct. And to drive home how even the simplest of mathematical concepts can propel us into deep waters, it ends with a list of currently unsolved problems. The authors have done an amazing job, and the world of recreational mathematics has a brilliant new jewel in its crown.

Preface

The British mathematician Ian Stewart pointed out in [395, p. 89] that “Mathematics intrigues people for at least three different reasons: because it is fun, because it is beautiful, or because it is useful.” Careful as mathematicians are, he wrote “at least”, and we would like to add (at least) one other feature, namely “surprising”. The Tower of Hanoi (TH) puzzle is a microcosmos of mathematics. It appears in different forms as a recreational game, thus fulfilling the fun aspect; it shows relations to Indian verses and Italian mosaics via its beautiful pictorial representation as an esthetic graph, it has found practical applications in psychological tests and its theory is linked with technical codes and phenomena in physics.

The authors are in particular amazed by numerous popular and professional (mathematical) books that display the puzzle on their covers. However, most of these books discuss only well-established basic results on the TH with incomplete arguments. On the other hand, in the last decades the TH became an object of numerous—some of them quite deep—investigations in mathematics, computer science, and neuropsychology, to mention just central scientific fields of interest. The authors have acted frequently as reviewers for submitted manuscripts on topics related to the TH and noted a lack of awareness of existing literature and a jumble of notation—we are tempted to talk about a Tower of Babel! We hope that this book can serve as a base for future research using a somewhat unified language.

More serious were the errors or mathematical myths appearing in manuscripts and even published papers (which did *not* go through our hands). Some “obvious assumptions” turned out to be questionable or simply wrong. Here is where many mathematical surprises will show up. Also astonishing are examples of how the mathematical model of a difficult puzzle, like the *Chinese rings*, can turn its solution into a triviality. A central theme of our book, however, is the meanwhile notorious *Frame-Stewart conjecture*, a claim of optimality of a certain solution strategy for what has been called *The Reve's puzzle*. Despite many attempts and even allegations of proofs, this had been[1] an open problem for more than 70 years.

Apart from describing the state of the art of its mathematical theory and applications, we will also present the historical development of the TH from its

[1]“has been” in the original preface

invention in the 19th century by the French number theorist Édouard Lucas. Although we are not professional historians of science, we nevertheless take historical remarks and comments seriously. During our research we encountered many errors or historical myths in literature, mainly stemming from the authors copying statements from other authors. We therefore looked into original sources whenever we could get hold of them.

Our guideline for citing other authors' papers was to include "the first and the best" (if these were two). The first, of course, means the first to our current state of knowledge, and the best means the best to our (current) taste.

This book is also intended to render homage to Édouard Lucas and one of his favorite themes, namely recreational mathematics in their role in mathematical education. The historical fact that games and puzzles in general and the TH in particular have demonstrated their utility is universally recognized (see, e.g., [383, 173]) more than 100 years after Lucas's highly praised book series started with [283].

Myths

Along the way we deal with numerous myths that have been created since the puzzle appeared on the market in 1883. These myths include mathematical misconceptions which turned out to be quite persistent, despite the fact that with a mathematically adequate approach it is not hard to clarify them entirely. A particular goal of this book is henceforth to act as a myth buster.

Prerequisites

A book of this size can not be fully self-contained. Therefore we assume some basic mathematical skills and do not explain fundamental concepts such as sets, sequences or functions, for which we refer the reader to standard textbooks like [156, 122, 370, 38, 398]. Special technical knowledge of any mathematical field is not necessary, however. Central topics of discrete mathematics, namely combinatorics, graph theory, and algorithmics are covered, for instance, in [270, 54], [432, 60, 104], and [247, 306], respectively. However, we will not follow notational conventions of any of these strictly, but provide some definitions in a glossary at the end of the book. Each term appearing in the glossary is put in **bold face** when it occurs for the first time in the text. This is mostly done in Chapter 0, which serves as a gentle introduction to ideas, concepts and notation of the central themes of the book. This chapter is written rather informally, but the reader should not be discouraged when encountering difficult passages in later chapters, because they will be followed by easier parts throughout the book.

The reader must also not be afraid of mathematical formulas. They shape the language of science, and some statements can only be expressed unambiguously when expressed in symbols. In a book of this size the finiteness of the number of symbols like letters and signs is a real limitation. Even if capitals and lower case, Greek and Roman characters are employed, we eventually run out of them. Therefore, in order to keep the resort to indices moderate, we re-use letters for sometimes quite different objects. Although a number of these are kept rather stable globally, like n for the number of discs in the TH or names of special sequences like Gros's g, many will only denote the same thing locally, e.g., in a section. We hope that this will not cause too much confusion. In case of doubt we refer to the indexes at the end of the book.

Algorithms

The TH has attracted the interest of computer scientists in recent decades, albeit with a widespread lack of rigor. This poses another challenge to the mathematician who was told by Donald Knuth in [245, p. 709] that "It has often been said that a person doesn't really understand something until he teaches it to someone else. Actually a person doesn't really understand something until he can teach it to a computer, i.e. express it as an algorithm." We will therefore provide provably correct algorithms throughout the chapters. Algorithms are also crucial for human problem solvers, differing from those directed to machines by the general human deficiency of a limited memory.

Exercises

Édouard Lucas begins his masterpiece *Théorie des nombres* [288, iii] with a (slightly corrected) citation from a letter of Carl Friedrich Gauss to Sophie Germain dated 30 April 1807 ("jour de ma naissance"): "Le goût pour les sciences abstraites, en général, et surtout pour les mystères des nombres, est fort rare; on ne s'en étonne pas. Les charmes enchanteurs de cette sublime science ne se décèlent dans toute leur beauté qu'à ceux qui ont le courage de l'approfondir."[2] Sad as it is that the first sentence is still true after more than 200 years, the second sentence, as applied to all of mathematics, will always be true. Just as it is impossible to get an authentic impression of what it means to stand on top of a sizeable mountain from reading a book on mountaineering without taking the effort to climb up oneself, a mathematics book has always to be read with paper and pencil in reach. The readers of our book are advised to solve the excercises posed throughout the chapters. They give additional insights into the topic, fill missing details, and challenge our skills. All exercises are addressed in the body of the text. They are of different grades of difficulty, but should be treatable at the place where they are cited. At least, they should then be *read*, because they may also contain new definitions and statements needed in the sequel. We collect hints and solutions to the problems at the end of the book, because we think that the reader has the right to know that the writers were able to solve them.

[2]"The taste for abstract sciences, in general, and in particular for the mysteries of numbers, is very rare; this doesn't come as a surprise. The enchanting charms of that sublime science do not disclose themselves in all their beauty but to those who have the courage to delve into it."

Contents

The book is organized into ten chapters. As already mentioned, Chapter 0 introduces the central themes of the book and describes related historical developments. Chapter 1 is concerned with the Chinese rings puzzle. It is interesting in its own right and leads to a mathematical model that is a prototype for an approach to analyzing the TH. The subsequent chapter studies the classical TH with three pegs. The most general problem solved in this chapter is how to find an optimal sequence of moves to reach an arbitrary regular state from another regular state. An important subproblem solved is whether the largest disc moves once or twice (or not at all). Then, in Chapter 3, we further generalize the task to reach a given regular state from an irregular one. The basic tool for our investigations is a class of graphs that we call *Hanoi graphs*. A variant of these, the so-called *Sierpiński graphs*, is introduced in Chapter 4 as a new and useful approach to Hanoi problems.

The second part of the book, starting from Chapter 5, can be understood as a study of variants of the TH. We begin with the famous The Reve's puzzle and, more generally, the TH with more than three pegs. The central role is played by the notorious Frame-Stewart conjecture which has been open since 1941. Computer experiments are also described that further indicate the inherent difficulty of the problem. We continue with a chapter in which we formally discuss the meaning of the notion of a variant of the TH. Among the variants treated we point out the *Tower of Antwerpen* and the *Bottleneck TH*. A special chapter is devoted to the *Tower of London*, invented in 1982 by T. Shallice, which has received an astonishing amount of attention in the psychology of problem solving and in neuropsychology, but which also gives rise to some deep mathematical statements about the corresponding *London graphs*. Chapter 8 treats TH type puzzles with oriented disc moves, variants which, together with the more-pegs versions, have received the broadest attention in mathematics literature among all TH variants studied.

In the final chapter we recapitulate open problems and conjectures encountered in the book in order to provide stimulation for those who want to pass their time expediently waiting for some Brahmins to finish a divine task.

Educational aims

With an appropriate selection from the material, the book is suitable as a text for courses at the undergraduate or graduate level. We believe that it is also a convenient accompaniment to mathematical circles. The numerous exercises should be useful for these purposes. Themes from the book have been employed by the authors as a leitmotif for courses in discrete mathematics, specifically by A. M. H. at the LMU Munich and in block courses at the University of Maribor and by S. K. at the University of Ljubljana. The playful nature of the subject lends itself to presentations of the fundamentals of mathematical thinking for a general audience. The TH was also at the base of numerous research programs for gifted students.

The contents of this book should, and we hope will, initiate further activities of this sort.

Feedback

If you find errors or misleading formulations, please send a note to the authors. Errata, sample implementations of algorithms, and other useful information will appear on the *TH-book* website at http://tohbook.info.

Acknowledgements

We are indebted to many colleagues and students who read parts of the book, gave useful remarks or kept us informed about very recent developments and to those who provided technical support. Especially we thank Jean-Paul Allouche, Jens-P. Bode, Drago Bokal, Christian Clason, Adrian Danek, Yefim Dinitz, Menso Folkerts, Rudolf Fritsch, Florence Gauzy, Katharina A. M. Götz, Andreas Groh, Robert E. Jamison, Marko Petkovšek, Amir Sapir, Marco Schwarz, Walter Spann, Arthur Spitzer, Sebastian Strohhäcker, Karin Wales, and Sara Sabrina Zemljič.

Throughout the years we particularly received input and advice from Simon Aumann, Daniele Parisse, David Singmaster, and Paul Stockmeyer (whose "list" [403] has been a very fruitful source).

Original photos were generously supplied by James Dalgety (The Puzzle Museum) and by Peter Rasmussen and Wei Zhang (Yi Zhi Tang Collection). For the copy of an important historical document we thank Claude Consigny (Cour d'Appel de Lyon). We are grateful to the Cnam – Musée des arts et métiers (Paris) for providing the photos of the original *Tour d'Hanoï*.

Special thanks go to the Birkhäuser/Springer Basel team. In particular, our Publishing Editor Barbara Hellriegel and Managing Director Thomas Hempfling guided us perfectly through all stages of the project for which we are utmost grateful to them, while not forgetting all those whose work in the background has made the book a reality.

A. M. H. wants to express his appreciation of the hospitality during his numerous visits in Maribor.

Last, but not least, we all thank our families and friends for understanding, patience, and support. We are especially grateful to Maja Klavžar, who, as a librarian, suggested to us that it was about time to write a comprehensive and widely accessible book on the Tower of Hanoi.

Andreas M. Hinz *München, Germany*
Sandi Klavžar *Trzin, Slovenia*
Uroš Milutinović *Maribor, Slovenia*
Ciril Petr *Hoče, Slovenia*

Preface to the Second Edition

In the preface to the first edition we wrote: "A central theme of our book [...] is the meanwhile notorious Frame-Stewart conjecture, a claim of optimality of a certain solution strategy for what has been called The Reve's puzzle. Despite many attempts and even allegations of proofs, this has been an open problem for more than 70 years." As it happens, in 2014 a historical breakthrough occurred when Thierry Bousch published a solution to The Reve's puzzle! His article is written in French and consequently less accessible to most researchers—especially since Bousch's ingenious proof is rather technical. We believe that this new development alone would have justified a second edition of this book, containing an English rendering of Bousch's approach.

Other significant progress happened since the first edition has been published in 2013. We emphasize here that Stockmeyer's conjecture concerning the smallest number of moves among all procedures that solve the Star puzzle, also listed among the open problems in Chapter 9 of the first edition, has been solved in 2017—again by Tierry Bousch. Some others of these open questions have been settled meanwhile. These solutions are addressed in the present edition. Moreover, extensive computer experiments on the Tower of Hanoi with more than three pegs have been performed in recent years. Other new material includes, e.g., the Tower of Hanoi with unspecified goal peg or with random moves and the Cyclic Tower of Antwerpen.

On our webpage http://tohbook.info twelve reviews of the first edition are referred to. We were pleased by their unanimous appreciation for our book. Specific remarks of the reviewers have been taken into account for the second edition. One desire of the readers was to find additional descriptions of some fundamental mathematical concepts, not to be found easily or satisfactory in the literature but used throughout the book. This guided us to extend Chapter 0 accordingly, e.g., by adding a section on sequences. This will make the book even more suitable as a textbook underlying mathematical seminars and circles which will also appreciate the more than two dozen new exercises.

When it comes to historical matters, an impressive account of the power of myths in math(s) is given in [304] (subtitle not arranged!). During our research for the new edition we found several much more recent legends that do not survive a meticulous reference to original sources which are given whenever we could get hold on them. In citing textbooks, we may not always refer to the most recent editions but to those which were at our disposal.

We extend our thanks to the individuals acknowledged in the preface of the first edition, among whom we want to re-emphasize Jean-Paul Allouche, Daniele Parisse, Sara Sabrina Zemljič, and Paul K. Stockmeyer for their constant support. Paul has been an inspiration for a number of new additions to the book, in particular demonstrating that the classical task can still offer new challenges. By presenting variants like his Star Tower of Hanoi he showed a vision for mathematically significant new topics.

While we were working on the second edition, the following people were of great help: Thierry Bousch, Jasmina Ferme, Brian Hayes, Andreas C. Höck, Caroline Holz auf der Heide, Richard Korf, Borut Lužar, and Michel Mollard.

Again we are grateful for the support from Birkhäuser. Clemens Heine and Luca Sidler guided us with patience and understanding through the jungle of nowadays scientific publishing.

A. M. H. wants to express his gratitude to the people of Slovenia for the opportunity to work on the topics of this book in the frame of the project *Exploration of the intrinsic structure of Tower graphs* supported by the Slovenian Research Agency (code J1-7110). These thanks go in particular to his co-authors and to Boštjan Brešar who stands for the hospitality of the University of Maribor.

Last but not least we thank Uroš Milutinović for inspiring discussions even after he decided, to our great sorrow, not to be a co-author of the second edition.

December 2017

Andreas M. Hinz — *München, Germany*
Sandi Klavžar — *Trzin, Slovenia*
Ciril Petr — *Hoče, Slovenia*

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