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Samir Adly

A Variational Approach to Nonsmooth Dynamics

Applications in Unilateral Mechanics
and Electronics

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A Murielle, Mélanie, Maxence et Fatna.

Foreword

During the French Revolution, the writer of a project of law on public instruction complained: *“The lack or scarcity of good elementary books has been, until now, one of the greatest obstacles in the way of better instruction. The reason for this scarcity is that, until now, scholars of great merit have almost always **preferred the glory of constructing the monument of science over the effort of lighting its entrance.**”* It is with this citation that, more than twenty-five years ago, I had wanted to begin the preface of the book *“Convex Analysis and Minimization Algorithms”* with my colleague Claude Lemarchal. It remains one of my favorite quotes, and I read it willingly on the occasion of the writing of this foreword.

Research in mathematics is rapidly advancing, in all directions, with an unrestrained literature produced by authors without taking the time to take stock of what has been done, what should be done or considered, in short, without “taking a break”. This is the role of survey-papers or books. There are, of course, specialized research books for professionals in the field, which allow visitors to visit all the rooms of the house (which is sometimes a huge building!), but also introductory books destined to enlighten the entrance to the building. The care taken in drafting the latter is all the greater because it is addressed to beginners, first visitors of the house, who would prefer not to get lost in the building, and whom it is necessary to guide. I believe that Springer’s series of Briefs responds partly to this objective.

That is why I suggested to Samir Adly, in a somewhat insistent way, to “write down neatly” the various advanced courses on the subject which I saw dispensed by him in several contexts and to various audiences. The author’s introduction, which comes just after this note, indicates well what the book is, without me having to talk about it here. The enthusiastic style of Samir Adly’s oral presentations, its exuberance all Mediterranean, are partly found here. So I wish good wind to this booklet, the best thing being that the winds of the Mediterranean Sea or of the Atlantic Ocean will lead young people to hoist the sails and to see how the entrance of the harbor or of the building are illuminated, before venturing further into it.

Toulouse, France
Spring 2017

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Preface

Nonsmooth dynamical systems are a class of evolution problems where “nonsmoothness” occurs. The word “nonsmooth,” like the words “nonconvex” or “nonlinear,” has to be understood in a general sense. If we agree, for example, that a function is smooth when it is continuously differentiable, that is a kind of first-order smoothness. In the same manner we can define p th-order smoothness ($p \in \mathbb{N}^*$) as the existence of derivatives up to order p along with their continuity. A *nonsmooth function* is a function which is not smooth. This vague term covers a very large class of functions such as nondifferentiable functions, discontinuous functions, multifunctions and beyond. Seen from this perspective, nonsmooth systems are those “bad” systems where a certain level of mathematical properties, traditionally used in classical smooth analysis, is not available. The classification of nonsmooth systems is, from that angle, not an easy task.

The aim of *nonsmooth analysis* is the development of an analog of differential calculus for functions which are not differentiable in the conventional sense, for sets which are not classical smooth manifolds (like sets with nonsmooth boundaries) and for set-valued maps (having values that might not be singletons). The following concepts have been introduced: generalized differentiation, subdifferentials, subgradients, generalized gradients, generalized directional derivatives, coderivatives as well as associated notions of tangent and normal cones to a set at a point. Nonsmooth analysis belongs to the broader domain named *variational analysis*. Generalized differentiation is at the core of modern variational analysis, a field initiated by J.-J. Moreau, R. T. Rockafellar, F. H. Clarke and enriched by many others.

Nonsmooth systems arise in many areas of research such as, for example, engineering, physics, biology, finance and economics. They are of great interest for the modeling of the dynamic behavior of many systems in concrete applications. For example in mechanical engineering, the contact of a rigid body with a foundation can be modeled by using nonsmooth constitutive laws with unilateral contact, friction and/or impact. The mathematical formulation of these problems leads naturally to nonsmooth dynamics. In control systems, nonsmoothness is usually used to model various problems such as sliding mode controllers, control with

hysteresis, and piecewise smooth systems. In electrical engineering, power converters involving nonsmooth electrical devices like ideal diodes, DIAC, transistors, DC-DC converters and amplifiers can also be modeled by nonsmooth dynamics. The mathematical formulation of many problems in Optimization and the Calculus of Variations involves inequality constraints and necessarily contains natural nonsmoothness (even if the data is smooth). In mathematical biology, nonsmooth characteristics are used for modeling gene regulatory or neural networks. One can truly say that our real world is intrinsically complex, nonlinear and nonsmooth. Our mathematical models should evolve in order to get as close as possible to the reality.

By nonsmooth dynamics we mean a system where the trajectory or the involved vector field is either continuous but not differentiable everywhere, or discontinuous, or is a set-valued map. In that case, access to the derivative is not possible, but we can use tools from modern variational analysis to handle the situation.

Among the mathematical issues that can be studied in the framework of nonsmooth dynamics, we may mention:

- (i) *Well-posedness*: definition of the concept of solutions, functional framework settings, existence and possible uniqueness of solutions, continuous dependence on initial data.
- (ii) *Stability Analysis*: Lyapunov stability, attractivity, invariance principles, periodic and quasi-periodic solutions, bifurcation analysis.
- (iii) *Control Analysis*: passivity, controllability, observability.
- (iv) *Numerical Analysis*: proposals of advanced numerical schemes for nonsmooth systems, proof of their convergence, their consistency, their efficiency, their robustness and their implementation.

The main purpose of this book is to show to engineers and researchers from **automatic control** and **applied mathematics** that advanced tools from optimization and variational analysis can be useful for the treatment of nonsmooth dynamical systems. Nonsmoothness is everywhere in real applications and nonsmooth dynamics provide a natural model for many phenomena in science and engineering. There is a big gap between abstract theoretical results developed by mathematicians and people in the socio-economic world who are interested in rigorous mathematical models only for their numerical simulation. Making these advanced mathematical tools accessible to engineers and nonspecialists is not an easy task. This book can be seen as a first step towards a scientific cultural transversality. Bridging the communities of *applied mathematicians*, *automatic controllers* and *engineers* has been one of the main objectives of our research in recent years.

This book considers mathematical models in nonsmooth mechanics (multibody dynamics with unilateral constraints, dry friction or impact) and nonregular electrical circuits (switching systems, relay, diodes, transistors and DC-DC converters). The model frameworks are: linear complementarity systems, evolution variational

inequalities, differential inclusions, measure differential inclusions, nonsmooth Lurie systems and Moreau's sweeping process.

In the first part, we establish the well-posedness of the problem (existence and uniqueness of the solution). The second part is devoted to the Lyapunov stability analysis of these systems. We give sufficient conditions under which the model is stable, asymptotically stable, finite-time stable or attractive. It is well-known in nonlinear systems and control theory that Lyapunov stability is of great importance due to its wide range of applications. The reason is that unstable systems are useless in practice and potentially dangerous. We show that what is known in smooth nonlinear systems could be adapted for nonsmooth systems by using advanced tools from convex and variational analysis. The last part of the book deals with the study of Moreau's sweeping process.

All the theoretical results that we developed are supported by concrete examples in nonsmooth mechanics and electronics as well as some numerical simulations. Due to the limited number of pages, however, we have not presented numerical methods for solving nonsmooth dynamical systems. We mention that [1, 2, 104] are relevant references for readers who are interested in that subject.

Chapter 1 is dedicated to the mathematical background that will be useful throughout the book.

Chapter 2 provides an overview of problems that can be studied in the framework of nonsmooth dynamics. We will particularly mention those problems that will not be treated in this book due to space limitations. The domain of nonsmooth dynamics is so vast and so used by different scientific communities that it is extremely difficult, if not impossible, to present a complete overview. We review piecewise dynamical systems, the Filippov concept of solutions for discontinuous differential equations, the notion of differential inclusions along with some general existence results, linear and nonlinear complementarity systems, evolution variational inequalities and their link with projected dynamical systems, and finally the so-called measure differential inclusions. The idea is to give the reader a quick but comprehensive snapshot of other classes of nonsmooth systems that can or cannot be captured by the models studied deeply in this book.

Chapter 3 focuses on the well-posedness and stability analysis (in the sense of Lyapunov) of first-order nonsmooth dynamics involving the subdifferential of a convex function. An existence and uniqueness result as well as sufficient conditions ensuring the stability, the asymptotic stability and the finite-time stability of this class of unilateral dynamics are given. An extension of LaSalle's invariance principle to this problem is also developed.

Chapter 4 treats the stability analysis of second-order nonsmooth dynamical systems with dry friction. This model plays an important role in unilateral mechanics where the motion of the system (with finite degrees of freedom) takes into account the unilaterality of the contact induced by friction forces. We give conditions on the data involved in the problem to ensure the existence and uniqueness of a solution. After reducing the problem to a first-order evolution variational inequality, we apply the results proved in Chap. 3 to analyze the stability, the asymptotic properties and the invariance principle for second-order

dynamics. Applications of the theoretical results are given to some examples in unilateral mechanics and nonregular electrical circuits. A rigorous mathematical stability analysis of a DC-DC Buck converter is also presented.

Chapter 5 is devoted to the study of nonsmooth Lurie dynamical systems. The well-posedness as well as the stability analysis of this class of differential inclusions are examined. The theoretical results are supplied with illustrations in power electronics.

Chapter 6 focuses on Moreau's sweeping processes. Existence and uniqueness results are given when the moving set of constraints is assumed to be convex and absolutely continuous or has a bounded retraction. A new variant of Moreau's sweeping process with velocity constraint in the moving set is also analyzed. This last class of problems subsumes as a particular case the evolution variational inequalities (widely used in applied mathematics and unilateral mechanics). Some applications of the sweeping process to a planning procedure economical model and to the modeling of nonregular electrical circuits are presented.

Since its future is closely linked to the technological developments in a wide variety of areas and sectors from the industrial world, we believe that the field of nonsmooth systems should grow rapidly in order to meet the major challenges of the 21st century. Many issues remain unresolved and need further investigation. As often in an area like scientific research, each answer brings its share of surprises and new challenges.

This monograph is the result of about twelve years of active research in the domain of nonsmooth dynamical systems.

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Samir Adly

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