

# Fundamentals of Probability and Stochastic Processes with Applications to Communications

Kun Il Park

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*To  
Sylvie*

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