Non-Instantaneous Impulses in Differential Equations

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Preface

Although impulsive systems were defined in the 1960s by Millman and Mishkis [94, 95], the theory of impulsive differential equations started its rapid development in the 1980s and continues to develop today. The development of the theory of impulsive differential equations gives an opportunity for some real-world processes and phenomena to be modeled more accurately. Impulsive equations are used for modeling in many different areas of science and technology (see, e.g., [46, 106]).

In the literature there are two popular types of impulses:

– *Instantaneous impulses*—the duration of these changes is relatively short compared to the overall duration of the whole process. The model is given by impulsive differential equations (see, e.g., monographs [59, 79, 104], and the cited therein bibliography).
– *Non-instantaneous impulses*—an impulsive action, which starts at an arbitrary fixed point and remains active on a finite time interval. E. Hernandez and D. O'Regan [56] introduced this new class of abstract differential equations where the impulses are not instantaneous, and they investigated the existence of mild and classical solutions.

In this book the impulses start abruptly at some points, and their actions continue on given finite intervals. As a motivation for the study of these systems, we consider the following simplified situation concerning the hemodynamical equilibrium of a person. In the case of a decompensation (e.g., high or low levels of glucose), one can prescribe some intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes, we can interpret the situation as an impulsive action which starts abruptly and stays active on a finite time interval. The model of this situation is the so-called non-instantaneous impulsive differential equation.

This book is the first published book devoted to the theory of differential equations with non-instantaneous impulses. A wide class of differential equations with non-instantaneous impulses are investigated, and these include:

– Ordinary differential equations with non-instantaneous impulses (scalar and n-dimensional case)
– Fractional differential equations with non-instantaneous impulses (with Caputo fractional derivatives of order $q \in (0, 1)$)
– Ordinary differential equations with non-instantaneous impulses occurring at random moments (with exponential, Erlang, or gamma distribution)

In Chapter 1 a systematic development of the theory of differential equations with non-instantaneous impulses is presented. In Section 1.2 some existence results are presented. In Section 1.3 stability theory is studied using modifications of Lyapunov's method. Classical continuous Lyapunov functions are commonly used for the qualitative investigation of different types of differential equations without impulses (see, e.g., the books [31, 70, 135]). Since the solutions of non-instantaneous impulsive equations are piecewise continuous functions, it is necessary to use appropriately defined piecewise continuous analogues of classical Lyapunov functions. It is noted that many authors apply piecewise continuous Lyapunov functions to study the stability of solutions of instantaneous impulsive equations (see the monographs [29, 79] and the cited therein bibliography). In Section 1.4 the monotone—iterative technique is applied to initial value problems for non-instantaneous impulsive equations. The main characteristic of our approximate method is the combination of the method of lower and upper solutions and an appropriate monotone method. These techniques are applied successfully to different types of differential equations without impulses (see, e.g., the book [74] and the cited therein references) and differential equations with instantaneous impulses (see, e.g., the book [59], and the cited therein references).

Chapter 2 is devoted to Caputo fractional differential equations with non-instantaneous impulses. In Section 2.1 the concepts of the presence of non-instantaneous impulses in Caputo fractional differential equations are given. Some existence results are presented. The basic stability theory to nonlinear fractional differential equations with non-instantaneous impulses by Lyapunov functions is developed. Several sufficient conditions for various types of stability for Caputo fractional derivatives are obtained. Also approximate methods for solving the initial value problem for fractional equations are developed.

In Chapter 3 non-instantaneous impulses starting at a random time and acting on an interval with initially given fixed length are studied. The p-exponential stability is defined and several sufficient conditions are given. We investigate both ordinary differential equations and Caputo fractional differential equations with random non-instantaneous impulses. The cases of exponentially, Erlang, and gamma distributed moments of the occurrence of impulses are studied.

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Contents

Introduction

Real-life processes and phenomena can be characterized by rapid changes in their state. The duration of these changes is relatively short compared to the overall duration of the whole process, and the changes turn out to be irrelevant to the development of the studied process. Mathematical models in such cases can be adequately created with the help of impulsive equations. Some examples of such processes can be found in physics, biology, population dynamics, ecology, pharmacokinetics, and others.

In the general case, impulsive equations consist of two parts:

– Differential equation that defines the continuous part of the solution;
– Impulsive part that defines the rapid change and the discontinuity of the solution

The first part of impulsive equations, that is, described by differential equations, could consist of ordinary differential equations, integro-differential equations, functional differential equations, partial differential equations, fractional differential equations, etc.

The second part of impulsive equations is called the *impulsive condition*. The points, at which the impulses occur, are called *moments of impulses*. The functions, that define the amount of the impulses are called *impulsive functions*. The time of action of the impulses, being small with respect to the whole duration of the studied process, can be negligibly small (instantaneous impulses), or the time could be an interval with a given length (non-instantaneous impulses). This leads to two basic types of impulsive equations:

– *Instantaneous impulses*—the duration of these changes is relatively short compared to the overall duration of the whole process. The model is given by impulsive differential equations (see, e.g., monographs [59, 79, 104], and the cited therein bibliography)

– *Non-instantaneous impulses*—an impulsive action, which starts at an arbitrary fixed point and remains active on a finite time interval. E. Hernandez and D. O'Regan [56] introduced this new class of abstract differential equations where the impulses are not instantaneous, and they investigated the existence of mild and classical solutions.

Chapter 1 discusses differential equations with non-instantaneous impulses. The systematic introduction to the concept of the solution of non-instantaneous impulses is provided. Many examples illustrate the topic. In Section 1.2 some existence results via Krasnoselskii fixed point theory are given. In Section 1.3.1 piecewise continuous scalar Lyapunov functions are applied to study stability, uniform stability, and asymptotic uniform stability of the solutions of nonlinear differential equations with non-instantaneous impulses. Several sufficient conditions for various types of stability are obtained, and the theoretical results are illustrated with several examples including a model in pharmacokinetics. In Section 1.3.2 the concept of practical stability as well as strict practical stability is generalized to nonlinear differential equations with non-instantaneous impulses. Stability and even asymptotic stability themselves are neither necessary nor sufficient to ensure practical stability. The desired state of a system may be mathematically unstable, but however, the system may oscillate sufficiently close to the desired state, and its performance is deemed acceptable. Practical stability is neither weaker nor stronger than the usual stability; an equilibrium can be stable in the usual sense, but not practically stable, and vice versa. Practical stability was studied for various types of differential equations (see, e.g., [34, 58, 60, 62, 63, 80, 91]). Also, the concept of strict stability (see, e.g., [6, 7, 78]) gives information on the boundedness of solutions. Section 1.4 considers an initial value problem for nonlinear non-instantaneous impulsive differential equation on a closed interval. The monotone iterative technique combined with the method of lower and upper solutions is applied to find approximately the solution of the given problems. A procedure for constructing two monotone functional sequences is given. The elements of these sequences are solutions of suitably chosen initial value problems for scalar linear non-instantaneous impulsive differential equations for which there is an explicit formula. Also, the elements of these sequences are lower/upper solutions of the given problem. We prove that both sequences converge and their limits are minimal and maximal solutions of the studied problem. An example, generalizing the logistic equation, is given to illustrate the procedure. Note that iterative techniques combined with lower and upper solutions are applied to approximately solve various problems for ordinary differential equations (see the classical monograph [74]), for various types of impulsive equations such as impulsive differential equations [28, 47], and for impulsive integro-differential equations [55], for impulsive differential equations with supremum [61].

Chapter 2 is devoted to Caputo fractional differential equations with non-instantaneous impulses. We study the case when the order of the fractional derivative $q \in (0, 1)$. In Section 2.1 two main concepts of solutions of fractional equations with non-instantaneous impulses are presented and illustrated with several examples. The statement of the problem is also discussed. Both approaches to the interpretation of solutions of non-instantaneous impulsive fractional differential equations are compared, and their advantages/disadvantages are discussed. In Section 2.2 some existence results are given. The concept of Ulam-type stability for Caputo fractional differential equations with non-instantaneous impulses is presented. Existence and Ulam-Hyers-Rassias stability results on a compact interval are proved for both types of interpretations of solutions of fractional differential equations with non-instantaneous impulses. The basic stability theory to nonlinear fractional differential equations with non-instantaneous impulses by Lyapunov function is developed in Section 2.3. Some comparison results applying Caputo fractional derivative, Dini fractional derivative, as well as Caputo fractional Dini derivative of Lyapunov functions are given. They are used to obtain and study various types of stability for nonlinear fractional differential equations with non-instantaneous impulses. In Section 2.3.3 the definition of Mittag-Leffler stability is extended to nonlinear fractional differential equations with non-instantaneous impulses using one of the approaches for the interpretation of solutions. Both the Caputo fractional derivative and the Caputo fractional Dini derivative are used to obtain some sufficient conditions for Mittag-Leffler stability with respect to non-instantaneous impulses. Stability, uniform stability, and asymptotic stability are studied in Section 2.3.4. Practical stability, uniform practical stability, practical quasi stability, as well as strong practical stability of the zero solution is the object of investigation in Section 2.3.5. In Section 2.3.6 strict stability is defined for nonlinear fractional differential equations with non-instantaneous impulses, and several criteria for such stability are obtained. This type of stability gives information concerning the rate of decay of solutions. All types of stability and most of the sufficient conditions are illustrated with examples. In Section 2.4 approximate methods for solving the initial value problem for fractional equations are developed. Both methods are based on the application of lower/upper solutions to nonlinear non-instantaneous impulsive fractional differential equations. An example illustrates the application of the procedure for constructing successive approximations.

In Chapter 3 non-instantaneous impulses starting at a random time are studied. Ordinary differential equations as well as Caputo fractional differential equations with random non-instantaneous impulses are studied. In Section 3.2 the p-exponential stability is defined for ordinary differential equations with random non-instantaneous impulses. Several sufficient conditions are given. The cases of exponentially (Section 3.2.1), Erlang (Section 3.2.2), and gamma (Section 3.2.3) distributed moments of the occurrence of impulses are studied. Section 3.3 discusses Caputo fractional differential equations with random non-instantaneous impulses exponentially distributed. The p-exponential stability is defined and sufficient conditions are obtained.