Introduction to Relation Algebras

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Relation Algebras, Volume 1

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To the memories of Alfred Tarski and Bjarni Jónsson.

Preface

The theory of relation algebras originated in the second half of the 1800s as a calculus of binary relations, in analogy with the calculus of classes (or unary relations) that was published around 1850 by George Boole in [15] and [16]. By 1900, it had developed into one of the three principal branches of mathematical logic. Yet, despite the intrinsic importance of the concepts occurring in this calculus, and their wide use and applicability throughout mathematics and science, as a mathematical discipline the subject fell into neglect after 1915.

It was revitalized and reformulated within an abstract axiomatic framework by Alfred Tarski in a seminal paper [104] from 1941, and since then the subject has grown substantially in importance and applicability. Numerous papers on relation algebras have been published since 1950, including papers in areas of computer science, and the subject has had a strong impact on such fields as universal algebra, algebraic logic, and modal logic. In particular, the study of relation algebras led directly to the development of a general theory of Boolean algebras with operators, an analogue for Boolean algebras of the well-known theory of groups with operators. This theory of Boolean algebras with operators appears to be especially well suited as an area for the application of mathematics to the theory and practice of computer science.

In my opinion, however, progress in the field and its application to other fields, and knowledge of the field among mathematicians, computer scientists, and philosophers, has been slowed substantially by the fact that, until recently, no systematic introductions to the subject existed. I believe that the appearance of such introductions will lead to a steady growth and influence of the theory and its applications, and to a much broader appreciation of the subject. It is for this reason that I have written these two volumes: to make the basic ideas and results of the subject (in this first volume), and some of the most important advanced areas of the theory (in the second volume, *Advanced Topics in Relation Algebras*), accessible to as broad an audience as possible.

Intended audience

This two-volume textbook is aimed at people interested in the contemporary axiomatic theory of binary relations. The intended audience includes, but is not limited to, graduate students and professionals in a variety of mathematical disciplines, logicians, computer scientists, and philosophers. It may well be that others in such diverse fields such as anthropology, sociology, and economics will also be interested in the subject. Kenneth Arrow, a Nobel Prize winning economist who in 1940 took a course on the calculus of relations with Tarski, said:

> It was a great course. ... the language of relations was immediately applicable to economics. I could express my problems in those terms.

The necessary mathematical preparation for reading this work includes mathematical maturity, something like a standard undergraduate-level course in abstract algebra, an understanding of the basic laws of Boolean algebra, and some exposure to naive set theory. Modulo this background, the text is largely self-contained. The basic definitions are carefully given and the principal results are all proved in some detail.

Each chapter ends with a historical section and a substantial number of exercises. In all, there are over 900 exercises. They vary in difficulty from routine problems that help readers understand the basic definitions and theorems presented in the text, to intermediate problems that extend or enrich the material developed in the text, to difficult problems that often present important results not covered in the text. Hints and solutions to some of the exercises are available for download from the Springer book webpage.

Readers of the first volume who are mainly interested in studying various types of binary relations and the laws governing these relations might want to focus their attention on Chapters 4 and 5, which deal with the laws and special elements. Those who are more interested in the algebraic aspects of the subject might initially skip Chapters 4 and 5, referring back to them later as needed, and focus more on Chapters 1–3, which concern the fundamental notions and examples of relation algebras, and Chapters 6–13, which deal with subalgebras, homomorphisms, ideals and quotients, simple and integral relation algebras, relativizations, direct products, weak and subdirect products, and minimal relation algebras respectively.

The second volume—which consists of Chapters 14–19—deals with more advanced topics: canonical extensions, completions, representations, representation theorems, varieties and universal classes, and atom structures. Readers who are principally interested in these more advanced topics might want to skip over most of the material in Chapters 4–13, and proceed directly to the material in the second volume that is of interest to them.

Acknowledgements

I took a fascinating course from Alfred Tarski on the theory of relation algebras in 1970, and my notes for that course have served as a framework for part of the first volume. I was privileged to collaborate with him over a ten-year period, and during that period I learned a great deal more about relation algebras, about mathematics in general, and about the writing of mathematics. The monograph [113] is one of the fruits of our collaboration. Without Tarski's influence, the present two volumes would not exist.

I am very much indebted to Hajnal Andréka, Robert Goldblatt, Ian Hodkinson, Peter Jipsen, Bjarni Jónsson, Richard Kramer, Roger Maddux, Ralph McKenzie, Don Monk, and István Németi for the helpful remarks and suggestions that they provided to me in correspondence during the composition of this work. Some of these remarks are referred to in the historical sections at the end of the chapters. In particular, Hajnal Andréka, István Németi, and I have had many discussions about relation algebras that have led to a close mathematical collaboration and friendship over more than thirty years. Gunther Schmidt and Michael Winter were kind enough to provide me with references to the literature concerning applications of the theory of relation algebras to computer science.

Savannah Smith read a draft of the first volume and called many typographic errors to my attention. Kexin Liu read the second draft of both volumes, caught numerous typographic errors, and made many

suggestions for stylistic improvements. Ian Hodkinson read through the final draft of the first volume, caught several typographic errors, and made a number of very perceptive and insightful recommendations. I am very grateful to all three of them.

Loretta Bartolini an editor of the mathematical series *Graduate Texts in Mathematics*, *Undergraduate Texts in Mathematics*, and *Universitext* published by Springer, has served as the editor for these two volumes. She has given me a great deal of advice and guidance during the publication process, and I am very much indebted to her and her entire production team at Springer for pulling out all stops, and doing the best possible job in the fastest possible way, to produce these two companion volumes. Any errors or flaws that remain in the volumes are, of course, my own responsibility.

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Introduction

Binary relations are a mathematical way of talking about relationships that exist between pairs of objects. Such relations permeate every aspect of mathematics, of science, and of human affairs. For example, the phrase *is a parent of* describes a binary relation that holds between two individuals a and b if and only if a is the father or the mother of b. Other examples of binary relations between human beings include the usual kinship relations such as *is an uncle of* or *is a sister of*, comparative relations such as *is richer than* or *is older than*, emotional relations such as *loves* or *is kind to*, material relations such as *owes money to* or *is a benefactor of*, and so on.

Just as there are natural operations on numbers, such as addition and multiplication, there are also natural operations on binary relations. For example, suppose F and M are the relations respectively described by the phrases *is a father of* and *is a mother of*. The union of F and M is the relation P described by the phrase *is a parent of*. The (relational) composition of F and M is the relation described by the phrase *is the maternal grandfather of*, while the composition of M and F is the relation described by the phrase *is the paternal grandmother of*. The converse of P is the relation described by the phrase *is a child of*. Thus, the operations of forming the union of two relations, the composition of two relations, and the converse of a relation are examples of operations on binary relations. Other examples include the operations of forming the intersection of two relations, the relational sum of two relations, and the complement of a relation. A set of binary relations that is closed under these operations constitutes a natural algebra of binary relations.

The operations on binary relations described in the preceding paragraph obey a great variety of laws. To give one example, let F, M, and P be the relations described above, and write B for the relation described by the phrase *is a brother of.* Suppose we first form the union $F \cup M$ of the relations F and M to obtain the relation P, and then we form the composition $B \mid P$ of B and P; the result is the relation described by the phrase *is an uncle of.* On the other hand, if we first form the compositions $B \mid F$ and $B \mid M$ to obtain the relations described by the phrases *is a paternal uncle of* and *is a maternal uncle of* respectively, and then form the union of $B \mid F$ and $B \mid M$, the result is again the relation *is an uncle of.* This state of affairs may be expressed in the form of an equation,

$$B \mid (F \cup M) = (B \mid F) \cup (B \mid M),$$

which is an instance of a law that is universally true for all binary relations, namely that the operation of forming compositions is distributive over the operation of forming unions in the sense that the composition of a relation R with the union of two relations S and T is equal to the union of the two compositions $R \mid S$ and $R \mid T$, in symbols

$$R \mid (S \cup T) = (R \mid S) \cup (R \mid T).$$

An example of a law that is not universally valid for all binary relations is the commutative law asserting that the operation of composition is commutative. Indeed, as we saw above, the composition of F and M is not the same as the composition of M and F, so the law

$$R \mid S = S \mid T$$

fails to hold for all binary relations R and S. The study of the laws that hold universally for all binary relations, that is to say, the laws that hold for all algebras of binary relations, formed an essential part of what is called the *calculus of relations* that was developed in the second half of the nineteenth century (see below).

The theory of relation algebras may be viewed as an abstract version of the calculus of relations. More precisely, it is an abstract algebraic theory of binary relations based on ten equational axioms that express laws true in all algebras of binary relations. The fundamental operations of the theory include abstract analogues of the Boolean operations of union, intersection, and complement (as applied to binary relations), and also abstract analogues of the Peircean operations of relational addition, relational multiplication or composition, and relational converse or inverse (the latter two are analogues for binary relations of the standard operations on functions of forming compositions and inverses of functions respectively). There are distinguished constants that are abstract analogues of the empty relation, the universal relation, the identity relation, and its complement, the diversity relation. (The adjectives "Boolean" and "Peircean" respectively honor George Boole, the founder of the calculus of classes, and Charles Sanders Peirce, the founder of the calculus of relations—see below.)

The arithmetic of the theory consists of the equations and implications between equations that are derivable from the axioms. These equations and implications express basic laws that govern the behavior of the fundamental operations on binary relations, and the mutual interactions of these operations. The expressive and deductive power of this arithmetic is incomparably richer than that of Boolean algebras. Most of the standard types of binary relations that proliferate every aspect of mathematics—reflexive relations, symmetric relations, transitive relations, equivalence relations, functions, injections, surjections, bijections, permutations, squares, rectangles, partial orders, linear orders, dense linear orders without endpoints, and so on—are definable in the theory of relation algebras in the sense that abstract analogues of these relations are definable; and the basic properties of these relations are all derivable from the abstract definitions and the axioms. Regarding this richness in means of expression and proof, Alfred Tarski [104] wrote in 1941:

> It may be noticed that many laws of the calculus of relations, and in particular the axioms. . . [we have adopted] resemble theorems of the theory of groups, relative multiplication playing the role of group-theoretic composition, and the converse of a relation corresponding to the group-theoretic inverse.. . . it turns out that the calculus of relations includes the elementary theory of groups and is, so to speak, a union of Boolean algebra and group theory. This fact accounts for the deductive power and mathematical richness of the calculus.

As Tarski was to show later, even more is true: the equational theory of relation algebras provides an adequate framework for the formalization of all of classical mathematics (see below).

The equational nature of the theory of relation algebras implies that the algebraic development of the subject is based on notions that are common to other branches of algebra such as group theory, lattice theory, and ring theory. These include the notions of subalgebra, homomorphism, congruence relation, quotient algebra, direct product, and subdirect product. Moreover, there are very close connections between the theory of relation algebras and several other well-known algebraic disciplines. For example, already in the early 1940s it was noticed by McKinsey (see [53]) that the algebra of complexes, or subsets, of an arbitrary group is always a relation algebra under the standard Boolean operations on subsets of the group and the Peircean operations induced on subsets by the operations of the group. The same is true of the algebra of subsets of a modular lattice with zero, as was shown by Maddux [73], and the algebra of subsets of a projective geometry, as was shown by Jónsson [49] and Lyndon [70].

Because the set of fundamental operations of a relation algebra includes the standard Boolean operations, the algebraic development of the theory of relation algebras also has close ties to algebraic and topological aspects of the theory of Boolean algebras. There are, for example, analogues for relation algebras of the various notions of complete extensions, such as canonical extensions and completions, that play such a significant role in Boolean algebra. There are important representation theorems that are analogues of Cayley's representation theorem for groups (every group is isomorphic to a group of permutations) and Stone's representation theorem for Boolean algebras (every Boolean algebra is isomorphic to a Boolean algebra of sets). In analogy with the well-known duality between Boolean algebras and Boolean topological spaces (compact, zero-dimensional, Hausdorff spaces), there are duality theorems establishing connections between relation algebras and various types of relational topological spaces (see, for example, [36]).

Finally, there are close connections to various systems of logic, including systems that are of interest to mathematicians, such as first-order logic, systems that are of interest to computer scientists, such as finite-variable logics, dynamic logics, and temporal logics, and systems that are of interest to philosophers, such as modal logics. Here are a few examples of such connections. (1) Every first-order theory of logic gives rise to a relation algebra of (congruence classes of) formulas in the same way as sentential logic gives rise to a Boolean algebra of (congruence classes of) formulas. (2) Hirsch-Hodkinson-Maddux [45] proved the following theorem about the number of variables needed to derive logically true sentences from the axioms of first-order logic, even in the case in which the sentences involved have just three variables: for every natural number $n \geq 4$, there is a logically-true first-order sentence Γ_n with exactly three distinct variables (each occurring many times) and exactly one binary relation symbol such that any proof of Γ_n based on the axioms of first-order logic requires n distinct variables in the sense that there is a proof of Γ_n that uses n distinct variables, and Γ_n cannot be proved with fewer than n distinct variables; the original proof of this theorem was carried out within the framework of the theory of relation algebras. (3) Tarski (see [113]) proved that first-order set-theory and first-order number theory can both be formalized in a very simple version of the equational theory of relation algebras in which there are no variables, quantifiers, or sentential connectives, and in which the only rule of inference is the high school rule of replacing equals by equals. He used this result to draw a number of surprising conclusions about logical systems and the foundations of mathematics; for example, there exist undecidable subsystems of sentential logic; also, all of classical mathematics can be developed in the framework of a version of a first-order system of set theory or number theory in which there are only three variables; thus, three variables suffice for expressing and proving all classical mathematical theorems. (Note that this result does not contradict the Hirsch-Hodkinson-Maddux result mentioned in (2), because the axioms of set theory and number theory are stronger than the axioms of first-order logic.)

The theory of relation algebras soon gave rise to a more general theory of Boolean algebras with operators—an analogue of the theory of groups with operators—that was created by Jónsson and Tarski in the late 1940s (see [54] and [55]). As it turns out, a surprising number of results that are initially proved in the context of relation algebras can be generalized to much broader classes of Boolean algebras with operators.

There are numerous applications of the calculus of relations and the theory of relation algebras to various areas of computer science, some of which are detailed in such books as [95] and [19], and in the whole series of proceedings of the RelMiCS/RAMiCS (Relational Methods in Computer Science/Relational and Algebraic Methods in Computer Science) conferences from 1993 onward. For instance, there are applications to databases (see, for example, [97] and [120]), and in particular there are close connections to the database language SQL, there are applications to program development and verification (see, for example, [10] and [30]), to programming semantics (see, for example, [9], [122], [97], and [77]), and to temporal and spatial reasoning (see, for example, [29] and [28]). There are also applications to other scientific disciplines, including anthropology (see, for example, [17] and [61]), economics (see, for example, [7] and [8]), linguistics (see, for example, [9]), social choice theory (see, for example, [7], [8], [79], and [96]), and voting theory (see, for example, [18]). For interesting applications of computer science to the study of relation algebras, see [56].

A brief history

The theory of relation algebras arose from the calculus of binary relations that was created in the second half of the nineteenth century by the English mathematician Augustus De Morgan, the American philosopher, scientist, and logician Charles Sanders Peirce, and the German mathematician Ernst Schröder as an analogue for binary relations of the calculus of classes (or unary relations) that was conceived by George Boole [15], [16], and refined by William Stanley Jevons. Concerning this early history of the subject, Tarski [104] wrote in 1941,

> The logical theory which is called the calculus of binary relations. . . has a strange and rather capricious line of historical development. Although some scattered remarks regarding the concept of relations are to be found already in the writings of medieval logicians, it is only with the last hundred years that this topic has become the subject of systematic investigation. The first beginnings of the contemporary theory of relations are to be found in the writings of A. De Morgan, who carried out extensive investigations in this domain in the fifties of the Nineteenth Century. De Morgan clearly recognized the inadequacy of traditional logic for the expression and justification, not merely of the more intricate arguments of mathematics and the sciences, but even of simple arguments occurring in every-day life; witness his famous aphorism, that all the logic of Aristotle does not permit us from the fact that a horse is an animal, to conclude that the head of a horse is the head of an animal. In his efforts to break the bonds of traditional logic and to expand the limits of logical inquiry, he directed his attention to the general concept of relations and fully recognized its significance. Nevertheless, De Morgan cannot be regarded as the creator of the modern theory of relations, since he did not possess an adequate apparatus for treating the subject in which he was interested, and was apparently unable to create such an apparatus. His investigations on relations show a lack of clarity and rigor which perhaps accounts for the neglect into which they fell in the following years.

> The title of creator of the theory of relations was reserved for C. S. Peirce. In several papers published between 1870 and 1882, he introduced and made precise all the fundamental concepts of the theory of relations and formulated and established its fundamental laws. Thus Peirce laid the foundation for the theory of relations as a deductive discipline; moreover he initiated the discussion of more profound problems in this domain. In particular, his investigations made it clear that a large part of the theory of relations can be presented as a calculus which is formally much like the calculus of classes developed by G. Boole and W. S. Jevons, but which greatly exceeds it in richness of expression and is therefore incomparably more interesting from the deductive point of view.
>
> Peirce's work was continued and extended in a very thorough and systematic way by E. Schröder. The latter's *Algebra und Logik der Relative*, which appeared in 1895 as the third volume of his *Vorlesungen über die Algebra der Logik*, is so far the only exhaustive account of the calculus of relations. At the same time, this book contains a wealth of unsolved problems, and seems to indicate the direction for future investigations.

By the beginning of the twentieth century, the subject had developed to the point where Bertrand Russell [92] could write in 1903:

> The subject of symbolic logic is formed by three parts: the calculus of propositions, the calculus of classes, and the calculus of relations.

The celebrated theorem of Leopold Löwenheim [65]—which is a cornerstone of modern mathematical logic, and which today would be formulated as stating that every formula valid in some model must in fact be valid in some countable model—was proved in the framework of the calculus of relations.

With the exception of Löwenheim, however, Peirce and Schröder did not have many followers. In the 1941 paper, after making the remarks quoted above, Tarski observed:

> It is therefore rather amazing that Peirce and Schröder did not have many followers. It is true that A. N. Whitehead and B. Russell, in *Principia mathematica*, included the theory of relations in the whole of logic, made this theory a central part of their logical system, and introduced many new and important concepts connected with the concept of relation. Most of these concepts do not belong, however, to the theory of relations proper but rather establish relations between this theory and other parts of logic: *Principia mathematica* contributed but slightly to the intrinsic development of the theory of relations as an independent deductive discipline. In general, it must be said that—though the significance of the theory of relations is universally recognized today—this

> theory, especially the calculus of relations, is now in practically the same stage of development as that in which it was forty-five years ago.

It was for this reason that Tarski [104] set out to revitalize and modernize the subject. He felt that there should be an axiomatic approach to the subject, a basic set of postulates from which all other laws could be derived using rules of inference. Moreover, the postulates and the methods of proof should only refer to relations, and not to other extraneous notions such as elements or pairs of elements of the universe of discourse. In other words, he imagined a presentation of the calculus of relations as an abstract algebraic discipline in much the same way as Louis Couturat and Edward Huntington had presented the Boole/Jevons calculus of classes as an abstract theory of Boolean algebras. He proposed a system of axioms—which he later simplified into ten equational axioms—and wrote that on the basis of these axioms he was practically sure he could derive all of the hundreds of laws to be found in Schröder's book. Nevertheless, he was unable to prove that every law true of all algebras of binary relations is derivable from these axioms, and so he posed this as his first problem, the completeness problem. He was also unable to show that every model of his set of axioms is isomorphic to a set relation algebra—that is to say, to an algebra of binary relations under the standard set-theoretically defined operations—and so he posed this as a second problem, the representation problem.

During the early 1940s, Tarski was able to prove that both set theory and number theory can be interpreted into a variable-free variant of his axiomatic theory of relation algebras, and therefore all of classical mathematics may be formalized within the framework of this variable-free equational theory (see [113]). On the basis of this interpretation, Tarski was also able to conclude that there is no mechanical procedure—no decision method—for establishing the truth or falsity of an equation in the theory of relation algebras. In this way, he was able to provide a concrete explanation for the observation made by Peirce regarding derivations in the calculus of relations that they "cannot be subjected to hard and fast rules like those of the Boolean calculus ...". Tarski pushed this result still further. He showed that his variable-free equational version of set theory can be interpreted as a finitely axiomatized subsystem of the two-valued sentential calculus. Hence, all of mathematics can be carried out within a subsystem of two-valued sentential logic.

A second phase in Tarski's work on the calculus of relations began around 1945. In that year, he held a seminar on relation algebras at the University of California at Berkeley. One portion of the seminar was devoted to a development of the arithmetic of the theory of relation algebras based on his axiom system. He and his student Louise Chin later published an important paper [23] on this subject. At the same time, Tarski attacked the completeness and representation problems with renewed energy. He focused on solving the representation problem. The first positive result was a quasi-representation theorem: every abstract relation algebra that is atomic is isomorphic to a relation algebra that is almost a set relation algebra; its universe consists of binary relations, and the basic operations of addition, relative multiplication, and converse have their set theoretic interpretation, but multiplication, relative addition, and complement do not. He showed further that if the atoms of the algebra are functions, then all of the operations have their set-theoretic interpretations, so we actually get a full representation of the abstract algebra as a set relation algebra, and not just a quasi-representation.

The quasi-representation theorem holds for atomic relation algebras, so the next logical step was to prove that every relation algebra $\mathfrak{A}$ can be extended to a complete and atomic relation algebra, the so-called the canonical (or perfect) extension of $\mathfrak{A}$. By the end of 1946 or the beginning of 1947, Tarski had succeeded in accomplishing this step, and he was also able to show that every Boolean algebra with additional unary distributive operations has a canonical extension that satisfies the same positive equations as the original algebra. He wrote about this to his former student, Bjarni Jónsson, who immediately became interested in the result, and worked to generalize it. In 1947, Jónsson succeeded in extending Tarski's theorem to classes of Boolean algebras with additional distributive operations of arbitrary ranks. In this way, the theory of Boolean algebras with operators was born—see the 1948 abstract [53], and the papers [54], and [55]. Today they play a rather important role in the applications of logic to computer science.

In the same 1948 abstract, Tarski observed that every relation algebra constructed from the complexes (or subsets) of a group is integral in the sense that it has no zero divisors with respect to the operation of relative multiplication, and he asked whether every integral relation algebra is isomorphic to an algebra of complexes of some group.

Around the end of 1948, Roger Lyndon—who had taken a course with Tarski in 1940—managed to construct a finite model of Tarski's axioms that is not isomorphic to a set relation algebra, and he simultaneously found an example of an equational law that is true in all set relation algebras but that is not derivable from Tarski's axioms. Thus, he solved negatively both of Tarski's first two problems (see [67]). A few years later, Tarski [110] was able to prove that the class of representable relation algebras—that is to say, the class of algebras isomorphic to set relations algebras—is indeed axiomatizable by a set of equations, but this set of equations may be infinite in number. Tarski asked whether there exists a finite set of equations that axiomatizes the class.

Lyndon's results began a new chapter in the theory of relation algebras: the study of non-representable algebras. Around 1958, Jónsson, who had experienced some difficulty in understanding Lyndon's construction of a non-representable relation algebra, found a way of constructing relation algebras from projective planes. If Desargues' theorem failed in the plane, then the relation algebra constructed from the plane would not be representable. Thus, Jónsson [49] was able to construct new examples of non-representable relation algebras.

Jónsson's construction was modified and extended by Lyndon [70] to a beautiful and simple construction of relation algebras from arbitrary projective geometries. For geometries of dimension one, Lyndon's construction yields particularly simple examples of non-representable relation algebras. In the same paper, Lyndon attacked Tarski's problem concerning integral relation algebras. He was able to make some progress on it, but he was unable to solve the problem completely.

Using the relation algebras constructed by Lyndon from projective lines, Tarski's former student Donald Monk [87] was able to show around 1964 that the class of representable relation algebras is not finitely axiomatizable. Thus, any system of equations axiomatizing this class of algebras is necessarily infinite in size. A couple of years later, Monk's student Ralph McKenzie proved in [82] (see also [83]) that Tarski's problem regarding integral relation algebras also has a negative solution: there are integral relation algebras that are not isomorphic to relation algebras constructed from complexes of a group. He went further by showing that the class of integral relation algebras that are isomorphic to such group complex algebras is not finitely axiomatizable, even over the class of integral, representable relation algebras.

The impetus given to the theory of relation algebras by Tarski's reformulation of the theory and by the interesting initial problems that he posed, the important results that were achieved in the solutions of these problems by himself and his students and grand-students Chin, Jónsson, Lyndon, Monk, and McKenzie, and the applications of these results to other domains such as computer science, have led to a revitalization of the calculus of relations in the form of the theory of relation algebras, and to the development of this subject into an active and ongoing field of research. Many new and interesting results have been obtained by scores of researchers from all parts of the globe.

We conclude this historical sketch by recalling the final sentences from Tarski's 1941 paper. They express an opinion that Tarski formed even before he obtained his major results concerning the theory of relation algebras, and it is an opinion that he kept to the end of his life.

> I do believe that the calculus of relations deserves much more attention than it receives. For, aside from the fact that the concepts occurring in this calculus possess an objective importance and are in these times almost indispensable in any scientific discussion, the calculus of relations has an intrinsic charm and beauty which makes it a source of intellectual delight to all who become acquainted with it.

Virtually everyone who has spent time working with relation algebras has learned to share Tarski's judgement.

Description and highlights of this volume

Chapter 1 describes the basic set-theoretical notions of the calculus of relations as conceived by Peirce [88]. The notion of a binary relation is introduced, examples of binary relations are given, and different ways of visualizing binary relations are presented. Some of the most common types of binary relations are discussed. The basic Boolean and Peircean operations on binary relations are defined and illustrated, and some examples of basic laws governing the behavior of these operations on binary relations are given. Boolean matrices are introduced, and the connections between the calculus of relations and the algebra of Boolean matrices are explored.

In Chapter 2, the notion of an abstract relation algebra is introduced on the basis of Tarski's ten equational axioms. The more general notion of a Boolean algebra with operators from Jónsson-Tarski [54] is defined and explored. The task of showing that a given Boolean algebra with operators is in fact a relation algebra is often non-trivial, so two theorems have been included that give necessary and sufficient criteria for an atomic Boolean algebra with operators to be a relation algebra.

Chapter 3 presents some of the classic examples of relation algebras. The most important of these is the class of set relation algebras, constructed as algebras of binary relations under the set-theoretically defined operations discussed in Chapter 1. In particular, full set relation algebras, consisting of all binary relations on a given set, play a special role in the development of the subject. A rather trivial, but important class of examples of relation algebras is constructed from Boolean algebras by using Boolean operations in the role of Peircean operations. Another class of examples is constructed by using congruence classes of formulas modulo first-order theories, under operations induced by the sentential connectives and the quantifiers of the logic. A highlight of the chapter is the careful examination of three types of relation algebras that arise as complex algebras—that is to say, algebras of subsets—of specific mathematical structures, under the Boolean operations of union and complement, and with Peircean operations that are defined in terms of the fundamental notions of the structures. The examples considered are the complex algebras of groups, the complex algebras of projective geometries, and the complex algebras of modular lattices with zero. A few examples of small relation algebras that are constructed in an ad hoc fashion from finite Boolean algebras with a small number of atoms (usually less than six in number) are also given. The chapter ends with a discussion of how algebraic models similar in structure to relation algebras may be used to demonstrate the independence of Tarski's ten axioms.

Chapter 4 contains a careful development of the general arithmetic of relation algebras—that is to say, a development of the basic laws governing the behavior of the operations on binary relations—on the basis of Tarski's ten axioms. As opposed to the arithmetic of Boolean algebras in which there is just one duality principle, namely the duality between addition and multiplication, in the arithmetic of relation algebras there are three different principles of duality at work: the first is a duality between left-hand and right-hand versions of each law; the second is a duality that arises when the Boolean operations of addition and multiplication are interchanged, and simultaneously the Peircean operations of relative addition and relative multiplication are interchanged in a given law; the third is the duality that arises as the composition of the first two dualities, that is to say, as a result of forming simultaneously the first and the second duals of a law. Thus, each law in the theory of relation algebras is closely associated with three other dual laws. The general laws of relation algebras have the flavor of a blending of laws from Boolean algebra and group theory, but their scope and content is much more complex than the laws from either Boolean algebra or group theory. For example, the semi-modular law

$$r \mathbin{;} (s \mathbin{+\!\!\!,} t) \leq (r \mathbin{;} s) \mathbin{+\!\!\!,} t$$

(in which $;$ and $\mathbin{+\!\!\!,}$ denote the abstract binary operations of relative multiplication and relative addition respectively) expresses in a compact equational form that an existential-universal statement implies a corresponding universal-existential statement, in much the same way as uniform continuity implies continuity.

Abstract versions of many of the important types of binary relations are studied in Chapter 5. These include symmetric elements, transitive elements, equivalence elements, ideal elements of various types, rectangles, squares, functions, bijections, and permutations. For example, an element r is defined to be an equivalence element just in case

$$r^{\smile} \leq r \qquad \text{and} \qquad r \mathbin{;} r \leq r$$

(where $^{\smile}$ denotes the operation of converse); these two inequalities respectively express in an abstract way the symmetry and transitivity of a relation denoted r. An element r is defined to be a function just in case $r^{\smile} \mathbin{;} r \leq 1'$ (where 1' is the identity element for relative multiplication); this inequality expresses in an abstract way that a relation denoted by r cannot map one element to two different elements. Several characterizations of each type of element are given, and the basic laws governing the behavior of the operations on these elements are established. There is a special emphasis on the distributive and modular laws that each type of element satisfies, and on the closure properties that sets of each type of element possess. For example, an element r is an equivalence element if and only if it satisfies the modular law

$$r \cdot [s \mathbin{;} (r \cdot t)] = (r \cdot s) \mathbin{;} (r \cdot t)$$

for all elements s and t (where $\cdot$ denotes the operation of multiplication); and r is a function if and only if it satisfies the distributive law

$$r ; (s \cdot t) = (r ; s) \cdot (r ; t)$$

for all elements s and t.

Chapter 6 develops the various notions that are connected with the concept of a subalgebra—subuniverses, subalgebras, complete subalgebras, regular subalgebras, elementary subalgebras, and sets of generators—and the properties that are preserved under the passage to various types of subalgebras. The highlights of the chapter include the following theorems. The subalgebras of a relation algebra form a complete, compactly generated lattice that is closed under directed unions. The regular subalgebras of atomic relation algebras are always atomic. The Atomic Subalgebra Theorem gives sufficient conditions on a subset W of a Boolean algebra with complete operators (and in particular, on a subset of a relation algebra) $\mathfrak{A}$ for the set of sums of subsets of W to be a regular, atomic subalgebra of $\mathfrak{A}$; as an example, this theorem is applied to the study of minimal relation algebras, and it is shown that every minimal relation algebra is necessarily finite. The downward and upward Löwenheim-Skolem-Tarski Theorems guarantee the existence of elementary subalgebras and elementary extensions of specified cardinalities. Finally, the union of a system of relation algebras directed by the relation of being an elementary subalgebra is shown to be an elementary extension of each algebra in the system.

Chapter 7 develops the various notions that are connected with the concept of a homomorphism—homomorphisms, epimorphisms, monomorphisms, isomorphisms, base isomorphisms, automorphisms, complete homomorphisms, and elementary embeddings—and properties that are preserved under the passage to homomorphic images. Highlights of the chapter include the following theorems. The Atomic Isomorphism Theorem gives necessary and sufficient conditions on a bijection φ between the sets of atoms of two complete and atomic Boolean algebras with complete operators $\mathfrak{A}$ and $\mathfrak{B}$ in order for φ to extend to a uniquely determined isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. There is also an analogue for monomorphisms. A version of the Exchange Principle (also known as the Exchange Theorem) from general algebra is proved; it says that if a relation algebra $\mathfrak{A}$ is embeddable into a relation algebra $\mathfrak{B}$ with certain properties, then $\mathfrak{A}$ is actually a subalgebra of a relation algebra $\mathfrak{C}$ that is isomorphic to $\mathfrak{B}$.

The notions of a congruence, an ideal, and the quotient of a relation algebra modulo a congruence or an ideal, are treated in Chapter 8. The discussion begins with congruences and lattices of congruences on a relation algebra, and then moves to ideals and lattices of ideals. The equivalence of the notions of a congruence and an ideal is established. It is shown that the ideals in a relation algebra form a complete, compactly generated, distributive lattice that is closed under directed unions. This theorem is followed by a discussion of the relationship between the ideals in a relation algebra $\mathfrak{A}$ and the Boolean ideals in the Boolean algebra of ideal elements in $\mathfrak{A}$. In particular, it is shown that the lattice of ideals in $\mathfrak{A}$ is isomorphic to the lattice of Boolean ideals in the Boolean algebra of ideal elements in $\mathfrak{A}$. Consequently, all of the results concerning the lattice of ideals in a relation algebra—for example, the existence of maximal ideals—may in principle be obtained as corollaries of the corresponding results for Boolean algebras.

Simple relation algebras—relation algebras in which there are exactly two ideals, the trivial ideal and the improper ideal—and integral relation algebras—non-degenerate relation algebras in which the relative product of two non-zero elements is always non-zero—are the topic of Chapter 9. A highlight of the chapter is the Simplicity Theorem, which states that, in contrast to the usual situation in, say, group theory or ring theory, the notion of simplicity for relation algebras is describable by a first-order universal sentence, and consequently every subalgebra of simple relation algebra is simple. Another important theorem says that every quantifier-free formula is equivalent to an equation in all simple relation algebras. The Integrality Theorem gives several characterizations of the notion of an integral relation algebra. One consequence of this theorem is that every integral relation algebra is simple. The chapter ends with a proof that for relation algebras, the notions of direct indecomposability, subdirect indecomposability, and simplicity all coincide.

Chapter 10 discusses the notion of the relativization of a relation algebra to an ideal element, and more generally to an equivalence element, and the properties that are preserved under the passage to relativizations. The main theorem of the chapter says that every quotient of a relation algebra $\mathfrak{A}$ modulo a principal ideal is isomorphic to a relativization of $\mathfrak{A}$ to an ideal element in $\mathfrak{A}$. Thus, quotients of $\mathfrak{A}$ modulo principal ideals have concrete representations as relativizations of $\mathfrak{A}$ and are therefore almost subalgebras of $\mathfrak{A}$ (up to isomorphisms).

The important topic of direct products and direct decompositions is treated in Chapter 11. The presentation is divided into two parts. The first part deals with the direct product of two relation algebras, and the second part with the direct product of arbitrary systems of relation algebras. Two types of direct decompositions of a relation algebra $\mathfrak{A}$ are discussed: the standard notion of an external direct decomposition, in which $\mathfrak{A}$ is shown to be isomorphic to the direct product of a system of relation algebras; and the notion of an internal direct decomposition, familiar from group theory, in which $\mathfrak{A}$ is shown to be equal to the internal product of a system of relativizations of $\mathfrak{A}$ to ideal elements. It is shown that for relation algebras, the two notions are essentially equivalent. One of the highlights of the chapter is the surprising Product Decomposition Theorem, which in its binary form says that a relation algebra $\mathfrak{A}$ is the (internal) direct product of relation algebras $\mathfrak{B}$ and $\mathfrak{C}$ if and only if there is an ideal element r in $\mathfrak{A}$ such that $\mathfrak{B}$ and $\mathfrak{C}$ are equal to the relativization of $\mathfrak{A}$ to r and the relativization of $\mathfrak{A}$ to the complement of r respectively. A more general version of this theorem is given that applies to direct products of arbitrary systems of relation algebras. Another highlight of the chapter is the Total Decomposition Theorem, which says that a relation algebra $\mathfrak{A}$ has a direct decomposition into a product of simple factors if and only if the Boolean algebra of ideal elements in $\mathfrak{A}$ is atomic and has the supremum property; and furthermore, if $\mathfrak{A}$ has a direct decomposition into simple factors, then this decomposition is unique up to permutations of the factors. One consequence of this theorem is that every complete and atomic relation algebra has a unique direct decomposition into a product of simple factors. In particular, every finite relation algebra has such a unique decomposition. Another consequence is the Complete Decomposition Theorem, which says that every complete relation algebra $\mathfrak{A}$ can be written in one and only one way as the internal direct product of relation algebras $\mathfrak{B}$ and $\mathfrak{C}$, where $\mathfrak{B}$ has a unique decomposition into the direct product of simple factors, and $\mathfrak{C}$ has no simple factors whatsoever. Examples of relation algebras without any simple factors are also given.

Other types of product constructions—in particular, weak direct products, ample direct products (which are subdirect products that are intermediate between weak direct products and full direct products), and subdirect products—are dealt with in Chapter 12. As in the case of direct products, there are external and internal versions of these products; and there are characterizations, in terms of systems of ideal elements, of when a relation algebra admits a decomposition using one of these products. For example, the Weak Product Decomposition Theorem says that a relation algebra $\mathfrak{A}$ is the weak internal product of a system of relation algebras $(\mathfrak{A}_i : i \in I)$ if and only if there is a system of ideal elements $(u_i : i \in I)$ partitioning the unit of $\mathfrak{A}$ such that the algebra $\mathfrak{A}_i$ coincides with the relativization of $\mathfrak{A}$ to u_i for each index i, and $\mathfrak{A}$ is generated by the union $\bigcup_{i\in I} A_i$ of the universes of the algebras in the system. The Semi-simplicity Theorem says that every relation algebra is isomorphic to a subdirect product of simple relation algebras. A consequence of this theorem is that an equation holds in all relation algebras if and only if it holds in all simple relation algebras.

The main goal of Chapter 13 is the classification of all minimal relation algebras. The notion of the type of a relation algebra is introduced (there are three: type one, type two, and type three), and it is shown that every simple relation algebra has a uniquely determined type. This is used to prove that the minimal simple relation algebras are, up to isomorphism, the three minimal set relation algebras $\mathfrak{M}_1$, $\mathfrak{M}_2$, and $\mathfrak{M}_3$ on sets of cardinality one, two, and three respectively. The Type Decomposition Theorem says that every relation algebra can be written as the internal product of three relation algebras of types one, two, and three respectively. The Classification Theorem for minimal relation algebras says that, up to isomorphism, there are exactly eight minimal relation algebras, namely the various possible direct products of the three minimal, simple relation algebras, with each factor occurring at most once. A more general classification theorem for relation algebras of types one and two is also given.

Chapters dependent on topics in the first volume

The material in Chapter 1 serves a motivational purpose, but it is also needed in Chapter 3 to understand two examples, set relation algebras and matrix algebras. The first of these examples plays a fundamental role throughout the two volumes, but especially in Chapters 16 and 17 in the second volume, which comprises Chapters 14–19. The remaining examples of Chapter 3 play an important role in Chapter 17.

The material in Section 2.1 is used throughout both volumes, while that of Section 2.2 is used in Chapters 6, 7, 14, 15, and 19. The material in Section 2.3 is used mainly in Chapter 3, while the logical terminology and notation of Section 2.4 is used consistently throughout both volumes, starting with Chapter 6, and that of Section 2.5 is used in one of the examples of Chapter 3 and in Chapters 16, 17, and 19.

The laws in Chapter 4 are used heavily in Chapter 5 and rather infrequently in other parts of the book. The special kinds of elements, and the laws about these elements, in Chapter 5 play a role in very specific parts of the book. Equivalence elements are needed in Chapter 10, and ideal elements are needed in Chapters 8–17. Functional elements are used in Chapter 17, rectangles in Chapter 18, and domains and ranges in Chapters 6, 9, 13, and 17.

The fundamental universal algebraic and Boolean algebraic notions (for example, the notions of a subalgebra, a regular subalgebra, a homomorphism, a complete homomorphism, an isomorphism, an ideal, quotient algebra, a simple algebra, a relativization of an algebra, and a direct product and a subdirect product of a system of algebras), and the basic results concerning these notions that are presented in various sections of Chapters 6–12 are used freely in subsequent chapters. The more specialized results in Section 6.6 are used in Chapters 7, 14, 17, and 18, while those in Section 6.8 are needed in Chapters 7, 16, 18, and 19. Similarly, the more specialized results in Section 7.6 are used in Chapters 14 and 17–19, while those in Section 7.7 are needed in Chapters 8 and 14–19, and those in Section 7.9 are needed in Chapter 18. Section 8.9 is used in Chapters 9, 11, 14, 15, and 17. The main results specific to relation algebras in Sections 9.1 and 9.2 are used in Chapters 10, 13, 14, 17, and 18, while those in Section 9.3 are needed in Chapter 11. Similarly, the main results specific to relation algebras in Section 10.4 are used in Chapters 11 and 13, while those in Section 10.5 are used in Chapters 14–16. Various of the direct product decomposition theorems for relation algebras that are given in Chapter 11 are used sporadically throughout all of the subsequent chapters of both volumes. The homomorphism decomposition theorems in Section 11.13 are used in Chapter 16. With regards to Chapter 12, only the main result of Section 12.3 is used elsewhere, namely in Chapters 13 and 16–18.

The material in the first three sections of Chapter 13 is needed in Chapter 18, while the material in the fourth section is not used again.

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