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Singular Limits in Thermodynamics of Viscous Fluids

Second Edition

Birkhäuser

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ISSN 2297-0320 ISSN 2297-0339 (electronic)
Advances in Mathematical Fluid Mechanics
ISBN 978-3-319-63780-8 ISBN 978-3-319-63781-5 (eBook)
DOI 10.1007/978-3-319-63781-5

Library of Congress Control Number: 2017949356

Mathematics Subject Classification (2010): 35Q30, 35-02, 76-02, 76N10, 80A22, 35B45, 76Q05

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Printed on acid-free paper

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The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface to the Second Edition

Besides the updates of the results discussed in the original version, the second edition of the book contains completely new material collected in Chaps. 7–9.

Chapter 7 has been considerably extended to problems involving slip boundary conditions on physical domains with "wavy" boundaries. In the almost incompressible regime, the boundary oscillates with an amplitude proportional and the frequency inversely proportional to the Mach number. The resulting effect on the motion is the same as for the non-slip boundary conditions; specifically the acoustic waves are damped and vanish in the low-Mach-number limit.

Chapter 8 has been essentially rewritten and amply extended by the new material. The singular limits are studied on a family of bounded domains that are large with respect to the characteristic speed of sound inversely proportional to the Mach number. Accordingly, in the low-Mach-number regime, the acoustic waves do not reach the boundary in a bounded lap of time, and the underlying acoustic system exhibits locally the same behavior as on an unbounded physical space. In particular, the dispersive estimates can be used to eliminate the acoustic component in the incompressible regime. This is illustrated by several examples, where the standard Strichartz estimates are used along with their "spectral" localization obtained by means of the celebrated RAGE theorem as well as its more refined version due to Tosio Kato. The theory is applied to the case of the limit passage from the compressible Navier-Stokes-Fourier system to the Boussinesq approximation.

Chapter 9 is completely new and extends the previous results to problems with vanishing dissipation–here represented by viscosity and heat conductivity of the fluid. Accordingly, the fluid becomes inviscid in the asymptotic limit, the motion being governed by a system of hyperbolic equations of Euler type. As a result, compactness provided by the presence of diffusive terms in the momentum and thermal energy equation is lost, and solutions of the primitive system are likely to develop oscillations in the course of the asymptotic limit. Still the problem enjoys a kind of structural stability encoded in the underlying system of equations. In particular, convergence to the target system can be recovered as long as the latter admits a regular solution. The "distance" between solutions of the primitive and target system is evaluated by a quantity termed *relative energy*. This approach, developed for hyperbolic systems of conservation laws by Constantine Dafermos, seems rather new in the context of viscous and heat-conducting fluids. Besides a rather elegant proof of convergence, this method gives rise to an explicit rate of convergence for certain model situations considered in Chap. 9.

The extended list of references includes the new results achieved since the first edition of the book was published as well as a piece of supplementary material relevant to the new topics addressed in the second edition.

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December 2016

Preface to the First Edition

Another advantage of a mathematical statement is
that it is so definite that it might be definitely wrong ...
Some verbal statements have not this merit.
L.F. Richardson (1881–1953)

Many interesting problems in mathematical fluid mechanics involve the behavior of solutions to systems of nonlinear partial differential equations as certain parameters vanish or become infinite. Frequently the solutions converge, provided the limit exists, to a solution of a limit problem represented by a qualitatively different system of differential equations. The simplest physically relevant example of this phenomenon is the behavior of a compressible fluid flow in the situation when the Mach number tends to zero, where the limit solution formally satisfies a system describing the motion of an incompressible fluid. Other interesting phenomena occur in the equations of magnetohydrodynamics, when either the Mach or the Alfven number or both tend to zero. As a matter of fact, most, if not all, mathematical models used in fluid mechanics rely on formal asymptotic analysis of more complex systems. The concept of incompressible fluid itself should be viewed as a convenient idealization of a medium in which the speed of sound dominates the characteristic velocity.

Singular limits are closely related to scale analysis of differential equations. Scale analysis is an efficient tool used both theoretically and in numerical experiments to reduce the undesirable and mostly unnecessary complexity of investigated physical systems. The simplified asymptotic limit equations may provide a deeper insight into the dynamics of the original mathematically more complicated system. They reduce considerably the costs of computations or offer a suitable alternative in the case when these fail completely or become unacceptably expensive when applied to the original problem. However, we should always keep in mind that these simplified equations are associated with singular asymptotic limits of the full governing equations, this fact having an important impact on the behavior of their solutions, for which degeneracies as well as other significant changes of the character of the governing equations become imminent.

Despite the vast amount of the existing literature, most of the available studies devoted to scale analysis are based on *formal* asymptotic expansion of (hypothetical) solutions with respect to one or several singular parameters. Although this might seem wasted or at least misguided effort from the purely theoretical point of view, such an approach proved to be exceptionally efficient in real-world applications. On the other hand, progress at the purely theoretical level has been hampered for many years by almost complete absence of a rigorous existence theory that would be applicable to the complex nonlinear systems arising in mathematical fluid dynamics. Although these problems are essentially well-posed on short time intervals or for small, meaning close to equilibrium states, initial data, a universal existence theory is still out of reach of modern mathematical methods. Still understanding the theoretical aspects of singular limits in systems of partial differential equations in general, and in problems of mathematical fluid mechanics in particular, is of great interest because of its immediate impact on the development of the theory. Last but not least, a rigorous identification of the asymptotic problem provides a justification of the mathematical model employed.

The concept of *weak solution* based on direct integral formulation of the underlying physical principles provides the only available framework for studying the behavior of solutions to problems in fluid mechanics in the large. The class of weak solutions is reasonably wide in order to accommodate all possible singularities that may develop in a finite time because of the highly nonlinear structures involved. Although optimality of this class of solutions may be questionable and still not completely accepted by the whole community, we firmly believe that the mathematical theory elaborated in this monograph will help to promote this approach and to contribute to its further development.

The book is designed as an introduction to problems of singular limits and scale analysis of systems of differential equations describing the motion of compressible, viscous, and heat-conducting fluids. Accordingly, the primitive problem is always represented by the Navier-Stokes-Fourier system of equations governing the time evolution of three basic state variables: the density, the velocity, and the absolute temperature associated to the fluid. In addition we assume the fluid is linearly viscous, meaning the viscous stress is determined through Newton's rheological law, while the internal energy flux obeys Fourier's law of heat conduction. The state equation is close to that of a perfect gas at least for moderate values of the density and the temperature. General ideas as well as the variational formulation of the problem based on a system of integral identities rather than partial differential equations are introduced and properly motivated in Chap. 1.

Chapters 2 and 3 contain a complete existence theory for the full Navier-Stokes-Fourier system without any essential restriction imposed on the size of the data as well as the length of the existence interval. The ideas developed in this part are of fundamental importance for the forthcoming analysis of singular limits.

Chapter 4 resumes the basic concepts and methods to be used in the study of singular limits. The underlying principle used amply in all future considerations is a decomposition of each quantity as a sum of its *essential* part relevant in the limit system and a *residual* part, where the latter admits uniform bounds induced by the available a priori estimates and vanishes in the asymptotic limit. This chapter also reveals an intimate relation between certain results obtained in this book and the so-called Lighthill's acoustic analogy used in numerous engineering applications.

Chapter 5 gives a comprehensive treatment of the low-Mach-number limit for the Navier-Stokes-Fourier system in the regime of low stratification, which means the Froude number is strongly dominated by the Mach number. As a limit system, we recover the well-known Oberbeck-Boussinesq approximation widely used in many applications. Remarkably, we establish uniform estimates of the set of weak solutions of the primitive system derived by help of the so-called dissipation inequality. This can be viewed as a direct consequence of the S*econd* l*aw of* t*hermodynamics* expressed in terms of the entropy balance equation, and the hypothesis of *thermodynamic stability* imposed on the constitutive relations. The convergence toward the limit system in the field equations is then obtained by means of the nowadays well-established technique based on compensated compactness. Another non-standard aspect of the analysis is a detailed description of propagation of the acoustic waves that arise as an inevitable consequence of *ill-prepared* initial data. In contrast with all previous studies, the underlying acoustic equation is driven by an external force whose distribution is described by a non-negative Borel measure. This is one of the intrinsic features encountered in the framework of weak solutions, where a piece of information concerning the energy transfer through possible singularities is lost.

Chapter 6 is primarily concerned with the strongly stratified fluids arising in astrophysics and meteorology. The central issue discussed here is the *anisotropy* of the sound wave propagation resulting from the strong stratification imposed by the gravitational field. Accordingly, the asymptotic analysis of the acoustic waves must be considerably modified in order to take into account the dispersion effects. As a model example, we identify the asymptotic system proposed by several authors as a suitable model of stellar radiative zones.

Most of the wave motions, in particular the sound wave propagation examined in this book, are strongly influenced by the effect of the boundary of the underlying physical space. If viscosity is present, a strong attenuation of the sound waves is expected at least in the case of so-called no-slip boundary conditions imposed on the velocity field. These phenomena are studied in detail in Chap. 7. In particular, it is shown that under certain geometrical conditions imposed on the physical boundary, the convergence of the velocity field in the low-Mach-number regime is strong, meaning free of time oscillations. Although our approach parallels other recent studies based on boundary layer analysis, we tried to minimize the number of necessary steps in the asymptotic expansion to make it relatively simple, concise, and applicable without any extra effort to a larger class of problems.

Another interesting aspect of the problem arises when singular limits are considered on large or possibly even unbounded spatial domains, where "large" is to be quantified with regard to the size of other singular dimensionless parameters. Such a situation is examined in Chap. 8. It is shown that the acoustic waves redistribute rapidly the energy and, leaving any fixed bounded subset of the physical space during a short time as the speed of sound becomes infinite, render the velocity field strongly (pointwise) convergent. Although the result is formally similar to those achieved in Chap. 7, the methods are rather different based on dispersive estimates of Strichartz type and finite speed of propagation for the acoustic equation.

Chapter 10 interprets the results on singular limits in terms of the acoustic analogies used frequently in numerical analysis. We identify the situations, where these methods are likely to provide reliable results and point out their limitations. Our arguments here rely on the uniform estimates obtained in Chap. 5.

The book is appended by two supplementary parts. In order to follow the subsequent discussion, the reader is recommended first to turn to the preliminary chapter, where the basic notation, function spaces, and other useful concepts, together with the fundamental mathematical theorems used in the book, are reviewed. The material is presented in a concise form and provided with relevant references when necessary. The appendix (Chap. 11) provides for reader's convenience some background material, with selected proofs, of more advanced but mostly standard results widely applicable in the mathematical theory of viscous compressible fluids in general and in the argumentation throughout this monograph in particular. Besides providing an exhaustive list of the relevant literature, the appendix is also aimed to offer a comprehensive and self-contained introduction to various specific recent mathematical tools designed to handle the problems arising in the mathematical theory of compressible fluids. As far as these results are concerned, the proofs are performed in full detail.

Since the beginning of this project, we have greatly profited from a number of seminal works and research studies. Although the most important references are included directly in the text of Chaps. 1–11, and Chap. 12 is designed to take the reader through the available literature on the topics addressed elsewhere in the book. In particular, a comprehensive list of the reference material is given, with a clear indication of the corresponding part discussed in the book. The reader is encouraged to consult these resources, together with the references cited therein, for a more complex picture of the problem as well as a more comprehensive exposition of some special topics.

The authors sincerely appreciate all who have offered comments and criticisms concerning the content of this book; in particular, many thanks go to Giovanni P. Galdi for many fruitful discussions and to Jan Březina, Petr Kaplický, Josef Málek, William Layton, Šárka Nečasová, Hana Petzeltová, Milan Pokorný, Dalibor Pražák, Jan Stebel, and Ivan Straškraba for careful reading of several chapters of the manuscript.

Praha, Czech Republic — Eduard Feireisl
Toulon, France — Antonín Novotný
September 2008

Acknowledgment

The work of Eduard Feireisl on the second edition of this monograph has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ERC Grant Agreement 320078. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

Notation, Definitions, and Function Spaces

1 Notation

Unless otherwise indicated, the symbols in the book are used as follows:

(i) The symbols const, c, c_i denote generic constants, usually found in inequalities. They do not have the same value as they are used at different parts of the text.

(ii) $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{C}$ are the sets of integers, positive integers, and complex numbers, respectively. The symbol $\mathbb{R}$ denotes the set of real numbers; $\mathbb{R}^N$ is the N-dimensional Euclidean space.

(iii) The symbol $\Omega \subset \mathbb{R}^N$ stands for a *domain*–an open connected subset of $\mathbb{R}^N$. The closure of a set $Q \subset \mathbb{R}^N$ is denoted by $\overline{Q}$; its boundary is ∂Q. By the symbol 1_Q, we denote the characteristic function of the set Q. The outer normal vector to ∂Q, if it exists, is denoted by $\mathbf{n}$.

The symbol $\mathcal{T}^N$ denotes the *flat torus*

$$\mathcal{T}^N = \left([-\pi, \pi]|_{\{-\pi;\pi\}}\right)^N = (\mathbb{R}|2\pi\mathbb{Z})^N$$

considered as a factor space of the Euclidean space $\mathbb{R}^N$, where $x \approx y$ whenever all coordinates of x differ from those of y by an integer multiple of 2π. Functions defined on $\mathcal{T}^N$ can be viewed as 2π-periodic in $\mathbb{R}^N$.

The symbol $B(a; r)$ denotes an (open) ball in $\mathbb{R}^N$ of center $a \in R^N$ and radius $r > 0$.

(iv) Vectors and functions ranging in a Euclidean space are represented by symbols beginning by a boldface minuscule, for example, $\mathbf{u}$, $\mathbf{v}$. Matrices (tensors) and matrix-valued functions are represented by special Roman characters such as $\mathbb{S}$ and $\mathbb{T}$; in particular, the identity matrix is denoted by $\mathbb{I} = \{\delta_{i,j}\}_{i,j=1}^N$. The symbol $\mathbb{I}$ is also used to denote the identity operator in a general setting.

The transpose of a square matrix $\mathbb{A} = \{a_{i,j}\}_{i,j=1}^N$ is $\mathbb{A}^T = \{a_{j,i}\}_{i,j=1}^N$. The trace of a square matrix $\mathbb{A} = \{a_{i,j}\}_{i,j=1}^N$ is $\text{trace}[\mathbb{A}] = \sum_{i=1}^N a_{i,i}$.

(v) The scalar product of vectors $\mathbf{a} = [a_1, \dots, a_N]$, $\mathbf{b} = [b_1, \dots, b_N]$ is denoted by

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{N} a_i b_i;$$

the scalar product of tensors $\mathbb{A} = \{A_{i,j}\}_{i,j=1}^{N}$, $\mathbb{B} = \{B_{i,j}\}_{i,j=1}^{N}$ reads

$$\mathbb{A} : \mathbb{B} = \sum_{i,j=1}^{N} A_{i,j} B_{j,i}.$$

The symbol $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product of vectors $\mathbf{a}$, $\mathbf{b}$; specifically

$$\mathbf{a} \otimes \mathbf{b} = \{\mathbf{a} \otimes \mathbf{b}\}_{i,j} = a_i b_j.$$

The vector product $\mathbf{a} \times \mathbf{b}$ is the antisymmetric part of $\mathbf{a} \otimes \mathbf{b}$. If $N = 3$, the vector product of vectors $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ is identified with a vector

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).$$

The product of a matrix $\mathbb{A}$ with a vector $\mathbf{b}$ is a vector $\mathbb{A}\mathbf{b}$ whose components are

$$[\mathbb{A}\mathbf{b}]_i = \sum_{j=1}^{N} A_{i,j} b_j \text{ for } i = 1, \dots, N,$$

while the product of a matrix $\mathbb{A} = \{A_{i,j}\}_{i,j=1}^{N,M}$ and a matrix $\mathbb{B} = \{B_{i,j}\}_{i,j=1}^{M,S}$ is a matrix $\mathbb{A}\mathbb{B}$ with components

$$[\mathbb{A}\mathbb{B}]_{i,j} = \sum_{k=1}^{M} A_{i,k} B_{k,j}.$$

(vi) The Euclidean norm of a vector $\mathbf{a} \in \mathbb{R}^N$ is denoted by

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{\sum_{i=1}^{N} a_i^2}.$$

The distance of a vector $\mathbf{a}$ to a set $K \subset \mathbb{R}^N$ is denoted as

$$\mathrm{dist}[\mathbf{a}, K] = \inf\{|\mathbf{a} - \mathbf{k}| \mid \mathbf{k} \in K\},$$

and the diameter of K is

$$\text{diam}[K] = \sup_{(x,y)\in K^2} |x - y|.$$

The closure of K is denoted by closure[K]; the Lebesgue measure of a set Q is $|Q|$.

2 Differential Operators

The symbol

$$\partial_{y_i} g(y) \equiv \frac{\partial g}{\partial y_i}(y),\ i = 1, \dots, N,$$

denotes the partial derivative of a function $g = g(y)$, $y = [y_1, \dots, y_N]$, with respect to the (real) variable y_i calculated at a point $y \in \mathbb{R}^N$. The same notation is used for distributional derivatives introduced below. Typically, we consider functions $g = g(t, x)$ of the *time variable* $t \in (0, T)$ and the *spatial coordinate* $x = [x_1, x_2, x_3] \in \Omega \subset \mathbb{R}^3$. We use italics rather than boldface minuscules to denote the independent variables although they may be vectors in many cases.

(i) The *gradient* of a scalar function $g = g(y)$ is a vector

$$\nabla g = \nabla_y g = [\partial_{y_1} g(y), \dots, \partial_{y_N} g(y)];$$

$\nabla^T g$ denotes the transposed vector to ∇g.

The *gradient* of a scalar function $g = g(t, x)$ with respect to the spatial variable x is a vector

$$\nabla_x g(t, x) = [\partial_{x_1} g(t, x), \partial_{x_2} g(t, x), \partial_{x_3} g(t, x)].$$

The *gradient* of a vector function $\mathbf{v} = [v_1(y), \dots, v_N(y)]$ is the matrix

$$\nabla \mathbf{v} = \nabla_y \mathbf{v} = \{\partial_{y_j} v_i\}_{i,j=1}^N;$$

$\nabla^T \mathbf{v}$ denotes the transposed matrix to $\nabla \mathbf{v}$. Similarly, the *gradient* of a vector function $\mathbf{v} = [v_1(t, x), v_2(t, x), v_3(t, x)]$ with respect to the space variables x is the matrix

$$\nabla_x \mathbf{v}(t, x) = \{\partial_{x_j} v_i(t, x)\}_{i,j=1}^3.$$

(ii) The *divergence* of a vector function $\mathbf{v} = [v_1(y), \dots, v_N(y)]$ is a scalar

$$\mathrm{div}\mathbf{v} = \mathrm{div}_y\mathbf{v} = \sum_{i=1}^{N} \partial_{y_i} v_i.$$

The *divergence* of a vector function depending on spatial and temporal variables $\mathbf{v} = [v_1(t,x), v_2(t,x), v_3(t,x)]$ with respect to the space variable x is a scalar

$$\mathrm{div}_x\mathbf{v}(t,x) = \sum_{i=1}^{3} \partial_{x_i} v_i(t,x).$$

The *divergence* of a tensor (matrix-valued) function $\mathbb{B} = \{B_{i,j}(t,x)\}_{i,j=1}^3$ with respect to the space variable x is a vector

$$[\mathrm{div}\mathbb{B}]_i = [\mathrm{div}_x\mathbb{B}(t,x)]_i = \sum_{j=1}^{3} \partial_{x_j} B_{i,j}(t,x),\ i = 1, \dots, 3.$$

(iii) The symbol $\Delta = \Delta_x$ denotes the *Laplace operator*,

$$\Delta_x = \mathrm{div}_x \nabla_x.$$

(iv) The *vorticity* (rotation) **curl** of a vectorial function $\mathbf{v} = [v_1(y), \dots, v_N(y)]$ is an antisymmetric matrix

$$\mathbf{curl}\ \mathbf{v} = \mathbf{curl}_y\mathbf{v} = \nabla\mathbf{v} - \nabla^T\mathbf{v} = \left\{\partial_{y_j} v_i - \partial_{y_i} v_j\right\}_{i,j=1}^{N}.$$

The vorticity of a vectorial function $\mathbf{v} = [v_1(t,x), \dots, v_3(t,x)]$ is an antisymmetric matrix

$$\mathbf{curl}_x\mathbf{v} = \nabla_x\mathbf{v} - \nabla_x^T\mathbf{v} = \left\{\partial_{x_j} v_i - \partial_{x_i} v_j\right\}_{i,j=1}^{3}.$$

The vorticity operator in $\mathbb{R}^3$ is sometimes interpreted as a vector $\mathbf{curl}\ \mathbf{v} = \nabla_x \times \mathbf{v}$.

(v) For a surface $\mathrm{S} \subset \mathbb{R}^3$, with an outer normal $\mathbf{n}$, we introduce the *normal gradient* of a scalar function $g : G \to \mathbb{R}^3$ defined on an open set $G \subset \mathbb{R}^3$ containing S as

$$\partial_{\mathbf{n}} g = \nabla_x g \cdot \mathbf{n}$$

and the tangential gradient as

$$[\partial_S]_i g = \partial_{x_i} g - (\nabla_x g \cdot \mathbf{n}) n_i, \ i = 1,2,3.$$

The *Laplace-Beltrami operator* on S is defined as

$$\Delta_s g = \sum_{i=1}^{3} [\partial_S]_i [\partial_S]_i g$$

(see Gilbarg and Trudinger [136, Chap. 16]).

3 Function Spaces

If not otherwise stated, all function spaces considered in this book are real. For a normed linear space X, we denote by $\| \cdot \|_X$ the *norm* on X. The duality pairing between an abstract vector space X and its dual X^* is denoted as $< \cdot ; \cdot >_{X^*;X}$, or simply $< \cdot ; \cdot >$ in case the underlying spaces are clearly identified in the context. In particular, if X is a Hilbert space, the symbol $< \cdot ; \cdot >$ denotes the scalar product in X.

The symbol span$\{M\}$, where M is a subset of a vector space X, denotes the space of all finite linear combinations of vectors contained in M.

(i) For $Q \subset \mathbb{R}^N$, the symbol $C(Q)$ denotes the set of continuous functions on Q. For a bounded set Q, the symbol $C(\overline{Q})$ denotes the Banach space of functions continuous on the closure $\overline{Q}$ endowed with norm

$$\|g\|_{C(\overline{Q})} = \sup_{y \in \overline{Q}} |g(y)|.$$

Similarly, $C(\overline{Q}; X)$ is the Banach space of vectorial functions continuous in $\overline{Q}$ and ranging in a Banach space X with norm

$$\|g\|_{C(\overline{Q})} = \sup_{y \in \overline{Q}} \|g(y)\|_X.$$

(ii) The symbol $C_{\text{weak}}(\overline{Q}; X)$ denotes the space of all vector-valued functions on $\overline{Q}$ ranging in a Banach space X continuous with respect to the weak topology. More specifically, $g \in C_{\text{weak}}(\overline{Q}; X)$ if the mapping $y \mapsto \|g(y)\|_X$ is bounded and

$$y \mapsto < f; g(y) >_{X^*;X}$$

is continuous on $\overline{Q}$ for any linear form f belonging to the dual space X^*.

We say that $g_n \to g$ in $C_{\mathrm{weak}}(\overline{Q}; X)$ if

$$< f; g_n >_{X^*;X} \to < f; g >_{X^*;X} \text{ in } C(\overline{Q}) \text{ for all } g \in X^*.$$

(iii) The symbol $C^k(\overline{Q})$, $Q \subset \mathbb{R}^N$, where k is a non-negative integer, denotes the space of functions on $\overline{Q}$ that are restrictions of k-times continuously differentiable functions on $\mathbb{R}^N$. $C^{k,\nu}(\overline{Q})$, $\nu \in (0,1)$ is the subspace of $C^k(\overline{Q})$ of functions having their k-th derivatives ν-Hölder continuous in $\overline{Q}$. $C^{k,1}(\overline{Q})$ is a subspace of $C^k(\overline{Q})$ of functions whose k-th derivatives are Lipschitz on $\overline{Q}$. For a bounded domain Q, the spaces $C^k(\overline{Q})$ and $C^{k,\nu}(\overline{Q})$, $\nu \in (0,1)$ are Banach spaces with norms

$$\|u\|_{C^k(\overline{Q})} = \max_{|\alpha| \le k} \sup_{x \in Q} |\partial^\alpha u(x)|$$

and

$$\|u\|_{C^{k,\nu}(\overline{Q})} = \|u\|_{C^k(\overline{Q})} + \max_{|\alpha|=k} \sup_{(x,y) \in Q^2, x \neq y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\nu},$$

where $\partial^\alpha u$ stands for the partial derivative $\partial_{x_1}^{\alpha_1} \dots \partial_{x_N}^{\alpha_N} u$ of order $|\alpha| = \sum_{i=1}^N \alpha_i$. The spaces $C^{k,\nu}(\overline{Q}; \mathbb{R}^M)$ are defined in a similar way. Finally, we set $C^\infty = \cap_{k=0}^\infty C^k$.

(iv)

■ ARZELÀ-ASCOLI THEOREM:

Theorem 1 *Let $Q \subset \mathbb{R}^M$ be compact and X a compact topological metric space endowed with a metric d_X. Let $\{v_n\}_{n=1}^\infty$ be a sequence of functions in $C(Q; X)$ that is equi-continuous, meaning for any $\varepsilon > 0$ there is $\delta > 0$ such that*

$$d_X\Big[v_n(y), v_n(z)\Big] \le \varepsilon \text{ provided } |y - z| < \delta \text{ independently of } n = 1, 2, \dots$$

Then $\{v_n\}_{n=1}^\infty$ is precompact in $C(Q; X)$, that is, there exists a subsequence (not relabeled) and a function $v \in C(Q; X)$ such that

$$\sup_{y \in Q} d_X\Big[v_n(y), v(y)\Big] \to 0 \text{ as } n \to \infty.$$

See Kelley [169, Chap. 7, Theorem 17]. □

(v) For $Q \subset \mathbb{R}^N$ an open set and a function $g : Q \to \mathbb{R}$, the symbol supp$[g]$ denotes the *support* of g in Q; specifically

$$\text{supp}[g] = \text{closure}\left[\{y \in Q \mid g(y) \neq 0\}\right].$$

(vi) The symbol $C_c^k(Q;\mathbb{R}^M)$, $k \in \{0, 1, \dots, \infty\}$ denotes the vector space of functions belonging to $C^k(Q;\mathbb{R}^M)$ and having compact support in Q. If $Q \subset \mathbb{R}^N$ is an open set, the symbol $\mathcal{D}(Q;\mathbb{R}^M)$ will be used alternatively for the space $C_c^\infty(Q;\mathbb{R}^M)$ endowed with the topology induced by the convergence:

$$\varphi_n \to \varphi \in \mathcal{D}(Q)$$

if

$$\text{supp}[\varphi_n] \subset K,\ K \subset Q \text{ a compact set},\ \varphi_n \to \varphi \text{ in } C^k(K) \text{ for any } k = 0, 1, \dots \tag{1}$$

We write $\mathcal{D}(Q)$ instead of $\mathcal{D}(Q;\mathbb{R})$.

The dual space $\mathcal{D}'(Q;\mathbb{R}^M)$ is the space of *distributions* on Ω with values in $\mathbb{R}^M$. Continuity of a linear form belonging to $\mathcal{D}'(Q)$ is understood with respect to the convergence introduced in (1).

(vii) A differential operator ∂^α of order $|\alpha|$ can be identified with a distribution

$$< \partial^\alpha v; \varphi >_{\mathcal{D}'(Q);\mathcal{D}(Q)} = (-1)^{|\alpha|} \int_Q v \partial^\alpha \varphi \, \mathrm{d}y$$

for any locally integrable function v.

(viii) The *Lebesgue spaces* $L^p(Q;X)$ are spaces of (Bochner) measurable functions v ranging in a Banach space X such that the norm

$$\|v\|^p_{L^p(Q;X)} = \int_Q \|v\|^p_X \, \mathrm{d}y \text{ is finite},\ 1 \leq p < \infty.$$

Similarly, $v \in L^\infty(Q;X)$ if v is (Bochner) measurable and

$$\|v\|_{L^\infty(Q;X)} = \operatorname*{ess\,sup}_{y \in Q} \|v(y)\|_X < \infty.$$

The symbol $L^p_{\text{loc}}(Q;X)$ denotes the vector space of locally L^p-integrable functions, meaning

$$v \in L^p_{\text{loc}}(Q;X) \text{ if } v \in L^p(K;X) \text{ for any compact set } K \text{ in } Q.$$

We write $L^p(Q)$ for $L^p(Q;\mathbb{R})$.

Let $f \in L^1_{loc}(Q)$ where Q is an open set. A *Lebesgue point* $a \in Q$ of f in Q is characterized by the property

$$\lim_{r \to 0+} \frac{1}{|B(a,r)|} \int_{B(a;r)} f(x) \mathrm{d}x = f(a). \tag{2}$$

For $f \in L^1(Q)$, the set of all Lebesgue points is of full measure, meaning its complement in Q is of zero Lebesgue measure. A similar statement holds for vector-valued functions $f \in L^1(Q; X)$, where X is a Banach space (see Brezis [40]).

If $f \in C(Q)$, then identity (2) holds for all points a in Q.

(ix)

■ LINEAR FUNCTIONALS ON $L^p(Q; X)$:

Theorem 2 *Let $Q \subset \mathbb{R}^N$ be a measurable set, X a Banach space that is reflexive and separable, and $1 \le p < \infty$.*

Then any continuous linear form $\xi \in [L^p(Q; X)]^*$ admits a unique representation $w_\xi \in L^{p'}(Q; X^*)$,

$$< \xi; v >_{L^{p'}(Q,X^*);L^p(Q;X)} = \int_Q < w_\xi(y); v(y) >_{X^*;X} \mathrm{d}y \text{ for all } v \in L^p(Q; X),$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Moreover the norm on the dual space is given as

$$\|\xi\|_{[L^p(Q;X)]^*} = \|w_\xi\|_{L^{p'}(Q;X^*)}.$$

Accordingly, the spaces $L^p(Q; X)$ are reflexive for $1 < p < \infty$ as soon as X is reflexive and separable.

See Gajewski et al. [130, Chap. IV, Theorem 1.14, Remark 1.9].

□

Identifying ξ with w_ξ, we write

$$[L^p(Q; \mathbb{R}^N)]^* = L^{p'}(Q; \mathbb{R}^N), \ \|\xi\|_{[L^p(Q;\mathbb{R}^N)]^*} = \|\xi\|_{L^{p'}(Q;\mathbb{R}^N)}, \ 1 \le p < \infty.$$

This formula is known as *Riesz representation theorem*.

(x) If the Banach space X in Theorem 2 is merely separable, we have

$$[L^p(Q; X)]^* = L^{p'}_{\text{weak}-(*)}(Q; X^*) \text{ for } 1 \le p < \infty,$$

where

$$L^{p'}_{\text{weak}-(*)}(Q; X^*)$$

$$:= \Big\{\xi : Q \to X^* \,\Big|\, y \in Q \mapsto <\xi(y); v>_{X^*;X} \text{ measurable for any fixed } v \in X,$$

$$y \mapsto \|\xi(y)\|_{X^*} \in L^{p'}(Q)\Big\}$$

(see Edwards [90] and Pedregal [231, Chap. 6, Theorem 6.14]).

(xi) *Hölder's inequality* reads

$$\|uv\|_{L^r(Q)} \le \|u\|_{L^p(Q)}\|v\|_{L^q(Q)}, \ \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

for any $u \in L^p(Q)$, $v \in L^q(Q)$, $Q \subset \mathbb{R}^N$ (see Adams [1, Chap. 2]).

(xii) *Interpolation inequality* for L^p-spaces reads

$$\|v\|_{L^r(Q)} \le \|v\|^{\lambda}_{L^p(Q)}\|v\|^{(1-\lambda)}_{L^q(Q)}, \ \frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q}, \ p < r < q, \ \lambda \in (0,1)$$

for any $v \in L^p \cap L^q(Q)$, $Q \subset \mathbb{R}^N$ (see Adams [1, Chap. 2]).

(xiii)

■ GRONWALL'S LEMMA:

Lemma 1 *Let* $a \in L^1(0,T)$, $a \ge 0$, $\beta \in L^1(0,T)$, $b_0 \in \mathbb{R}$, *and*

$$b(\tau) = b_0 + \int_0^\tau \beta(t)\,\mathrm{d}t$$

be given. Let $r \in L^\infty(0,T)$ *satisfy*

$$r(\tau) \le b(\tau) + \int_0^\tau a(t)r(t)\,\mathrm{d}t \text{ for a.a. } \tau \in [0,T].$$

Then

$$r(\tau) \le b_0 \exp\left(\int_0^\tau a(t)\,\mathrm{d}t\right) + \int_0^\tau \beta(t)\exp\left(\int_t^\tau a(s)\,\mathrm{d}s\right)\,\mathrm{d}t$$

for a.a. $\tau \in [0,T]$.

See Carroll [49]. □

4 Sobolev Spaces

(i) A domain $\Omega \subset \mathbb{R}^N$ is of class $\mathcal{C}$ if for each point $x \in \partial\Omega$, there exist $r > 0$ and a mapping $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$ belonging to a function class $\mathcal{C}$ such that–upon rotating and relabeling the coordinate axes if necessary–we have

$$\left.\begin{aligned} \Omega \cap B(x;r) &= \{y \mid \gamma(y') < y_N\} \cap B(x,r) \\ \partial\Omega \cap B(x;r) &= \{y \mid \gamma(y') = y_N\} \cap B(x,r) \end{aligned}\right\}, \quad \text{where } y' = (y_1, \dots, y_{N-1}).$$

In particular, Ω is called *Lipschitz domain* if γ is Lipschitz.

If $A \subset \Gamma := \partial\Omega \cap B(x;r)$, γ is Lipschitz, and $f : A \to R$, then one can define the surface integral

$$\int_A f \,\mathrm{d}S_x := \int_{\Phi_\gamma(A)} f(y', \gamma(y')) \sqrt{1 + \sum_{i=1}^{N-1} \left(\frac{\partial\gamma}{\partial y_i}\right)^2} \,\mathrm{d}y',$$

where $\Phi_\gamma : \mathbb{R}^N \mapsto \mathbb{R}^N$, $\Phi_\gamma(y', y_N) = (y', y_N - \gamma(y'))$, whenever the (Lebesgue) integral at the right-hand side exists. If $f = 1_A$, then $S_{N-1}(A) = \int_A \mathrm{d}S_x$ is the surface measure on $\partial\Omega$ of A that can be identified with the $(N-1)$-Hausdorff measure on $\partial\Omega$ of A (cf. Evans and Gariepy [97, Chap. 4.2]). In the general case of $A \subset \partial\Omega$, one can define $\int_A f \,\mathrm{d}S_x$ using a covering $\mathcal{B} = \{B(x_i;r)\}_{i=1}^M$, $x_i \in \partial\Omega$, $M \in \mathbb{N}$ of $\partial\Omega$ by balls of radii r and subordinated partition of unity $\mathcal{F} = \{\varphi_i\}_{i=1}^M$ and set

$$\int_A f \,\mathrm{d}S_x = \sum_{i=1}^M \int_{\Gamma_i} \varphi_i f \,\mathrm{d}S_x, \quad \Gamma_i = \partial\Omega \cap B(x_i;r);$$

see Nečas [219, Sect. I.2] or Kufner et al. [175, Sect. 6.3].

A Lipschitz domain Ω admits the outer normal vector $\mathbf{n}(x)$ for a.a. $x \in \partial\Omega$. Here *a.a.* refers to the surface measure on $\partial\Omega$.

The distance function $d(x) = \mathrm{dist}[x, \partial\Omega]$ is Lipschitz continuous. In addition, d is differentiable a.a. in $\mathbb{R}^3$, and

$$\nabla_x d(x) = \frac{x - \xi(x)}{d(x)}$$

whenever d is differentiable at $x \in \mathbb{R}^3 \setminus \Omega$, where ξ denotes the nearest point to x on $\partial\Omega$ (see Ziemer [277, Chap. 1]). Moreover, if the boundary $\partial\Omega$ is of class C^k, then d is k-times continuously differentiable in a neighborhood of $\partial\Omega$, provided $k \geq 2$ (see Foote [127]).

We say that a family of domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ is *uniformly of class* $\mathcal{C}$ if the radius r of the balls $B(x,r)$ as well as the norm of the functions γ in $\mathcal{C}$ describing the boundary can be taken the same for all $\varepsilon > 0$.

(ii) The *Sobolev spaces* $W^{k,p}(Q;\mathbb{R}^M)$, $1 \leq p \leq \infty$, k a positive integer, are the spaces of functions having all distributional derivatives up to order k in $L^p(Q;\mathbb{R}^M)$. The norm in $W^{k,p}(Q;\mathbb{R}^M)$ is defined as

$$\|\mathbf{v}\|_{W^{k,p}(Q;\mathbb{R}^M)} = \left\{ \begin{array}{l} \left(\sum_{i=1}^M \sum_{|\alpha|\leq k} \|\partial^\alpha v_i\|^p_{L^p(Q)} \right)^{1/p} \text{ if } 1 \leq p < \infty \\ \\ \max_{1\leq i\leq M,\, |\alpha|\leq k} \{\|\partial^\alpha v_i\|_{L^\infty(Q)}\} \text{ if } p = \infty \end{array} \right\},$$

where the symbol ∂^α stands for any partial derivative of order $|\alpha|$.

If Q is a bounded domain with boundary of class $C^{k-1,1}$, then there exists a continuous linear operator which maps $W^{k,p}(Q)$ to $W^{k,p}(\mathbb{R}^N)$; it is called *extension operator*. If, in addition, $1 \leq p < \infty$, then $W^{k,p}(Q)$ is separable and the space $C^k(\overline{Q})$ is its dense subspace.

The space $W^{1,\infty}(Q)$, where Q is a bounded domain, is isometrically isomorphic to the space $C^{0,1}(\overline{Q})$ of Lipschitz functions on $\overline{Q}$.

For basic properties of Sobolev functions, see Adams [1] or Ziemer [277].

(iii) The symbol $W_0^{k,p}(Q;\mathbb{R}^M)$ denotes the completion of $C_c^\infty(Q;\mathbb{R}^M)$ with respect to the norm $\|\cdot\|_{W^{k,p}(Q;\mathbb{R}^M)}$. In what follows, we identify $W^{0,p}(\Omega;R^N) = W_0^{0,p}(\Omega;\mathbb{R}^N)$ with $L^p(\Omega;\mathbb{R}^N)$.

We denote $\dot{L}^p(Q) = \{u \in L^p(Q) \mid \int_Q u \,\mathrm{d}y = 0\}$ and $\dot{W}^{1,p}(Q) = W^{1,p}(Q) \cap \dot{L}^p(Q)$. If $Q \subset \mathbb{R}^N$ is a bounded domain, then $\dot{L}^p(Q)$ and $\dot{W}^{1,p}(Q)$ can be viewed as closed subspaces of $L^p(Q)$ and $W^{1,p}(Q)$, respectively.

(iv) Let $Q \subset \mathbb{R}^N$ be an open set, $1 \leq p \leq \infty$, and $v \in W^{1,p}(Q)$. Then we have:

(a) $|v|^+, |v|^- \in W^{1,p}(Q)$ and

$$\partial_{x_j}|v|^+ = \left\{ \begin{array}{c} \partial_{x_k} v \text{ a.a. in } \{v > 0\} \\ \\ 0 \text{ a.a. in } \{v \leq 0\} \end{array} \right\},\ \partial_{x_j}|v|^- = \left\{ \begin{array}{c} \partial_{x_k} v \text{ a.a. in } \{v < 0\} \\ \\ 0 \text{ a.a. in } \{v \geq 0\} \end{array} \right\},$$

$j = 1, \dots, N$, where $|v|^+ = \max\{u, 0\}$ denotes a positive part and $|v|^- = \min\{u, 0\}$ a negative part of v.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function and $f \circ v \in L^p(Q)$, then $f \circ v \in W^{1,p}(Q)$ and

$$\partial_{x_j}[f \circ v](x) = f'(v(x))\partial_{x_j} v(x) \text{ for a.a. } x \in Q.$$

For more details, see Ziemer [277, Sect. 2.1].

(v) *Dual spaces to Sobolev spaces.*

■ DUAL SOBOLEV SPACES:

Theorem 3 *Let $\Omega \subset \mathbb{R}^N$ be a domain, and let $1 \leq p < \infty$. Then the dual space $[W_0^{k,p}(\Omega)]^*$ is a proper subspace of the space of distributions $\mathcal{D}'(\Omega)$. Moreover, any linear form $f \in [W_0^{k,p}(\Omega)]^*$ admits a representation*

$$< f; v >_{[W_0^{k,p}(\Omega)]^*; W_0^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \int_\Omega (-1)^{|\alpha|} w_\alpha \, \partial^\alpha v \, \mathrm{d}x, \tag{3}$$

$$\textit{where } w_\alpha \in L^{p'}(\Omega), \ \frac{1}{p} + \frac{1}{p'} = 1.$$

The norm of f in the dual space is given as

$$\|f\|_{[W_0^{k,p}(\Omega)]^*} = \begin{cases} \inf \left\{ \left(\sum_{|\alpha| \leq k} \|w_\alpha\|^{p'}_{L^{p'}(\Omega)} \right)^{1/p'} \middle| w_\alpha \textit{ satisfy } (3) \right\} \\ \textit{for } 1 < p < \infty; \\ \inf \left\{ \max_{|\alpha| \leq k} \{ \|w_\alpha\|_{L^\infty(Q)} \} \middle| w_\alpha \textit{ satisfy } (3) \right\} \\ \textit{if } p = 1. \end{cases}$$

The infimum is attained in both cases.

See Adams [1, Theorem 3.8] and Mazya [209, Sect. 1.1.14]. □

The dual space to the Sobolev space $W_0^{k,p}(\Omega)$ is denoted as $W^{-k,p'}(\Omega)$.

The dual to the Sobolev space $W^{k,p}(\Omega)$ admits formally the same representation formula as (3). However it cannot be identified as a space of distributions on Ω. A typical example is the linear form

$$< f; v > = \int_\Omega \mathbf{w}_f \cdot \nabla_x v \, \mathrm{d}x, \ \text{with } \mathrm{div}_x \mathbf{w}_f = 0$$

that vanishes on $\mathcal{D}(\Omega)$ but generates a non-zero linear form when applied to $v \in W^{1,p}(\Omega)$.

(vii)

■ RELLICH-KONDRACHOV EMBEDDING THEOREM:

Theorem 4 *Let $\Omega \subset R^N$ be a bounded Lipschitz domain.*

(i) Then, if $kp < N$ and $p \geq 1$, the space $W^{k,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any

$$1 \leq q \leq p^* = \frac{Np}{N - kp}.$$

Moreover, the embedding is compact if $k > 0$ and $q < p^*$.

(ii) If $kp = N$, the space $W^{k,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [1, \infty)$.

(iii) If $kp > N$, then $W^{k,p}(\Omega)$ is continuously embedded in $C^{k-[N/p]-1,\nu}(\overline{\Omega})$, where $[\,]$ denotes the integer part and

$$\nu = \begin{cases} [\frac{N}{p}] + 1 - \frac{N}{p} \text{ if } \frac{N}{p} \notin \mathbb{Z}, \\ \text{arbitrary positive number in } (0, 1) \text{ if } \frac{N}{p} \in \mathbb{Z}. \end{cases}$$

Moreover, the embedding is compact if $0 < \nu < [\frac{N}{p}] + 1 - \frac{N}{p}$.

See Ziemer [277, Theorem 2.5.1, Remark 2.5.2]. □

The symbol $\hookrightarrow$ will denote continuous embedding; $\hookrightarrow\hookrightarrow$ indicates compact embedding.

The following result may be regarded as a direct consequence of Theorem 4.

■ EMBEDDING THEOREM FOR DUAL SOBOLEV SPACES:

Theorem 5 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $k > 0$ and $q < \infty$ satisfy*

$$q > \frac{p^*}{p^* - 1}, \text{ where } p^* = \frac{Np}{N - kp} \text{ if } kp < N,$$

$$q > 1 \text{ for } kp = N,$$

or

$$q \geq 1 \text{ if } kp > N.$$

Then the space $L^q(\Omega)$ is compactly embedded into the space $W^{-k,p'}(\Omega)$, $1/p + 1/p' = 1$.

(viii) The *Sobolev-Slobodeckii spaces* $W^{k+\beta,p}(Q)$, $1 \leq p < \infty$, $0 < \beta < 1$, $k = 0, 1, \ldots$, where Q is a domain in $\mathbb{R}^L$, $L \in \mathbb{N}$, are Banach spaces of functions with finite norm

$$W^{k+\beta,p}(Q) = \left(\|v\|^p_{W^{k,p}(Q)} + \sum_{|\alpha|=k} \int_Q \int_Q \frac{|\partial^\alpha v(y) - \partial^\alpha v(z)|^p}{|y-z|^{L+\beta p}} \, \mathrm{d}y \, \mathrm{d}z \right)^{\frac{1}{p}};$$

see, e.g., Nečas [219, Sect. 2.3.8].

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Referring to the notation introduced in **(i)**, we say that $f \in W^{k+\beta,p}(\partial\Omega)$ if $(\varphi f) \circ (\mathbb{I}', \gamma) \in W^{k+\beta,p}(\mathbb{R}^{N-1})$ for any $\Gamma = \partial\Omega \cap B$ with B belonging to the covering $\mathcal{B}$ of $\partial\Omega$ and φ the corresponding term in the partition of unity $\mathcal{F}$. The space $W^{k+\beta,p}(\partial\Omega)$ is a Banach space endowed with an equivalent norm $\|\cdot\|_{W^{k+\beta,p}(\partial\Omega)}$, where

$$\|v\|^p_{W^{k+\beta,p}(\partial\Omega)} = \sum_{i=1}^{M} \|(v\varphi_i) \circ (\mathbb{I}', \gamma)\|^p_{W^{k+\beta,p}(\mathbb{R}^{N-1})}.$$

In the above formulas, $(\mathbb{I}', \gamma) : \mathbb{R}^{N-1} \to \mathbb{R}^N$ maps y' to $(y', \gamma(y'))$. For more details, see, e.g., Nečas [219, Sect. 3.8].

In the situation when $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, the Sobolev-Slobodeckii spaces admit similar embeddings as classical Sobolev spaces. Namely, the embeddings

$$W^{k+\beta,p}(\Omega) \hookrightarrow L^q(\Omega) \text{ and } W^{k+\beta,p}(\Omega) \hookrightarrow C^s(\overline{\Omega})$$

are compact provided $(k+\beta)p < N$, $1 \le q < \frac{Np}{N-(k+\beta)p}$, and $s = 0, 1, \ldots, k$, $(k-s+\beta)p > N$, respectively. The former embedding remains continuous (but not compact) at the border case $q = \frac{Np}{N-(k+\beta)p}$.

(ix)

■ Trace Theorem for Sobolev Spaces and Green's formula:

Theorem 6 *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain.*

Then there exists a linear operator γ_0 with the following properties:

$$[\gamma_0(v)](x) = v(x) \text{ for } x \in \partial\Omega \text{ provided } v \in C^\infty(\overline{\Omega}).$$

$$\|\gamma_0(v)\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \le c\|v\|_{W^{1,p}(\Omega)} \text{ for all } v \in W^{1,p}(\Omega).$$

$$\ker[\gamma_0] = W^{1,p}_0(\Omega)$$

provided $1 < p < \infty$.

Conversely, there exists a continuous linear operator

$$\ell : W^{1-\frac{1}{p},p}(\partial\Omega) \to W^{1,p}(\Omega)$$

such that

$$\gamma_0(\ell(v)) = v \text{ for all } v \in W^{1-\frac{1}{p},p}(\partial\Omega)$$

provided $1 < p < \infty$.

In addition, the following formula holds:

$$\int_\Omega \partial_{x_i} uv \, \mathrm{d}x + \int_\Omega u \partial_{x_i} v \, \mathrm{d}x = \int_{\partial\Omega} \gamma_0(u)\gamma_0(v) n_i \, \mathrm{dS}_x, \ i = 1, \ldots, N,$$

for any $u \in W^{1,p}(\Omega)$, $v \in W^{1,p'}(\Omega)$, *where* $\mathbf{n}$ *is the outer normal vector to the boundary* $\partial\Omega$.

See Nečas [219, Theorems 5.5, 5.7]. □

The dual $[W^{1-\frac{1}{p},p}(\partial\Omega)]^*$ to the Sobolev-Slobodeckii space $W^{1-\frac{1}{p},p}(\partial\Omega)) = W^{\frac{1}{p'},p}(\partial\Omega)$ is denoted simply by $W^{-\frac{1}{p'},p'}(\partial\Omega)$.

(ix) If $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, then we have the interpolation inequality

$$\|v\|_{W^{\alpha,r}(\Omega)} \leq c\|v\|^{\lambda}_{W^{\beta,p}(\Omega)}\|v\|^{1-\lambda}_{W^{\gamma,q}(\Omega)}, \ 0 \leq \lambda \leq 1, \tag{4}$$

for

$$0 \leq \alpha, \beta, \gamma \leq 1, \ 1 < p, q, r < \infty, \ \alpha = \lambda\beta + (1-\lambda)\gamma, \ \frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q}$$

(see Sects. 2.3.1, 2.4.1, and 4.3.2 in Triebel [257]).

(x)

■ SOBOLEV INEQUALITY:

Theorem 7 *Let* $N \geq 2$ *and* $1 < p, q < \infty$ *such that*

$$p < N, \ q = \frac{Np}{N-p}.$$

There is a constant $c = c(p, q, N)$ *such that*

$$\|v\|_{L^q(\mathbb{R}^N)} \leq c\|\nabla_x v\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)} \text{ for any } v \in C^\infty_c(\mathbb{R}^N).$$

See Ziemer [277, Chap. 2, Theorem 2.4.1]. □

The Sobolev spaces $D^{1,p}(\Omega)$ are defined as the completion of functions $C_c^\infty(\Omega)$ with respect to the gradient (pseudo)-norm

$$\|v\|_{D^{1,p}(\Omega)} = \|\nabla_x v\|_{L^p(\Omega;R^N)}.$$

If Ω is a regular *bounded* domain, the spaces $D^{1,p}(\Omega)$ coincide with $W_0^{1,p}(\Omega)$ via Poincaré's inequality. If Ω is unbounded, we get, in view of Theorem 7,

$$D^{1,p}(\Omega) \subset L^q(\Omega) \text{ if } \Omega \subset \mathbb{R}^N,\ p < N,\ q = \frac{Np}{N-p}. \tag{5}$$

In general, the spaces $D^{1,p}$ consist of classes of functions differing by an additive constant; see Galdi [131].

(xi)

■ EXTENSION PROPERTY:

Theorem 8 *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then there exists an extension operator E,*

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N),\ 1 \le p < \infty,$$

with the following properties:

- *$E[v]$ is compactly supported for any $v \in W^{1,p}(\Omega)$.*
- *$E[v]|_\Omega = v$.*
-

$$\|E[v]\|_{W^{1,p}(\mathbb{R}^N)} \le c\|v\|_{W^{1,p}(\Omega)},$$

where the constant depends only on p and on the radius and the Lipschitz constant of the charts describing $\partial\Omega$.

See Stein [251, Chap. 6]. □

5 Fourier Transform

Let $v = v(x)$ be a complex-valued function integrable on $\mathbb{R}^N$. The *Fourier transform* of v is a complex-valued function $\mathcal{F}_{x\to\xi}[v]$ of the variable $\xi \in \mathbb{R}^N$ defined as

$$\mathcal{F}_{x\to\xi}[v](\xi) = \left(\frac{1}{2\pi}\right)^{N/2} \int_{\mathbb{R}^N} v(x)\exp(-\mathrm{i}\xi\cdot x)\,\mathrm{d}x. \tag{6}$$

Therefore, the Fourier transform $\mathcal{F}_{x\to\xi}$ can be viewed as a continuous linear mapping defined on $L^1(\mathbb{R}^N)$ with values in $L^\infty(\mathbb{R}^N)$.

(i) For u, v complex-valued square-integrable functions on $\mathbb{R}^N$, we have *Parseval's identity*:

$$\int_{\mathbb{R}^N} u(x)\overline{v}(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} \mathcal{F}_{x\to\xi}[u](\xi)\overline{\mathcal{F}_{x\to\xi}[v]}(\xi)\,\mathrm{d}\xi,$$

where bar denotes the complex conjugate. Parseval's identity implies that $\mathcal{F}_{x\to\xi}$ can be extended as a continuous linear mapping defined on $L^2(\mathbb{R}^N)$ with values in $L^2(\mathbb{R}^N)$.

(ii) The symbol $\mathcal{S}(\mathbb{R}^N)$ denotes the space of smooth rapidly decreasing (complex-valued) functions; specifically, $\mathcal{S}(\mathbb{R}^N)$ consists of functions u such that

$$\sup_{|\alpha|\leq m}\sup_{x\in\mathbb{R}^N}(1+|x|^2)^s|\partial^\alpha u| < \infty$$

for all $s, m = 0, 1, \ldots$. We say that $u_n \to u$ in $\mathcal{S}(\mathbb{R}^N)$ if

$$\sup_{|\alpha|\leq m}\sup_{x\in\mathbb{R}^N}(1+|x|^2)^s|\partial^\alpha(u_n-u)| \to 0,\ \ s, m = 0, 1, \ldots. \tag{7}$$

The space of *tempered distributions* is identified as the dual $\mathcal{S}'(\mathbb{R}^N)$. Continuity of a linear form belonging to $\mathcal{S}'(\mathbb{R}^N)$ is understood with respect to convergence introduced in (7).

The Fourier transform introduced in (6) can be extended as a bounded linear operator defined on $\mathcal{S}(\mathbb{R}^N)$ with values in $\mathcal{S}(\mathbb{R}^N)$. Its inverse reads

$$\mathcal{F}^{-1}_{\xi\to x}[f] = \left(\frac{1}{2\pi}\right)^{N/2}\int_{\mathbb{R}^N} f(\xi)\exp(ix\cdot\xi)\mathrm{d}\xi. \tag{8}$$

(iii) The Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^N)$ is defined as

$$< \mathcal{F}_{x\to\xi}[f]; g > = < f; \mathcal{F}_{x\to\xi}[g] > \text{ for any } g \in \mathcal{S}(\mathbb{R}^N). \tag{9}$$

It is a continuous linear operator defined on $\mathcal{S}'(\mathbb{R}^N)$ onto $\mathcal{S}'(\mathbb{R}^N)$ with the inverse $\mathcal{F}^{-1}_{\xi\to x}$,

$$< \mathcal{F}^{-1}_{\xi\to x}[f], g > = < f, \mathcal{F}^{-1}_{\xi\to x}[g] >, \quad f \in \mathcal{S}'(R^N), g \in \mathcal{S}(\mathbb{R}^N). \tag{10}$$

(iv) We recall formulas

$$\partial_{\xi_k}\mathcal{F}_{x\to\xi}[f] = \mathcal{F}_{x\to\xi}[-\mathrm{i}x_k f], \quad \mathcal{F}_{x\to\xi}[\partial_{x_k} f] = \mathrm{i}\xi_k\mathcal{F}_{x\to\xi}[f], \tag{11}$$

where $f \in \mathcal{S}'(\mathbb{R}^N)$, and

$$\mathcal{F}_{x\to\xi}[f * g] = \Big(\mathcal{F}_{x\to\xi}[f]\Big) \times \Big(\mathcal{F}_{x\to\xi}[g]\Big), \tag{12}$$

where $f \in \mathcal{S}(\mathbb{R}^N)$, $g \in \mathcal{S}'(\mathbb{R}^N)$, and $*$ denotes *convolution*.

(v) A partial differential operator D of order m,

$$D = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha,$$

can be associated to a *Fourier multiplier* in the form

$$\tilde{D} = \sum_{|\alpha|\le m} a_\alpha (\mathrm{i}\xi)^\alpha, \ \xi^\alpha = \xi_1^{\alpha_1} \dots \xi_N^{\alpha_N}$$

in the sense that

$$D[v](x) = \mathcal{F}^{-1}_{\xi\to x}\left[\sum_{|\alpha|\le m} a_\alpha (\mathrm{i}\xi)^\alpha \mathcal{F}_{x\to\xi}[v](\xi)\right], \ v \in \mathcal{S}(\mathbb{R}^N).$$

The operators defined through the right-hand side of the above expression are called *pseudodifferential operators*.

(vi) Various *pseudodifferential operators* used in the book are identified through their Fourier symbols:

- *Riesz transform*:

$$\mathcal{R}_j \approx \frac{-i\xi_j}{|\xi|}, \ j = 1, \dots, N.$$

- *Inverse Laplacian*:

$$(-\Delta)^{-1} \approx \frac{1}{|\xi|^2}.$$

- *The "double" Riesz transform*:

$$\{\mathcal{R}\}_{i,j=1}^N, \ \mathcal{R} = \Delta^{-1}\nabla_x \otimes \nabla_x, \ \mathcal{R}_{i,j} \approx \frac{\xi_i \xi_j}{|\xi|^2}, \ i,j = 1, \dots, N.$$

- *Inverse divergence*:

$$\mathcal{A}_j = \partial_{x_j} \Delta^{-1} \approx \frac{\mathrm{i}\xi_j}{|\xi|^2}, \ j = 1, \dots, N.$$

We denote

$$\mathbb{A} : \mathcal{R} \equiv \sum_{i,j=1}^{3} A_{i,j}\mathcal{R}_{i,j}, \ \mathcal{R}[\mathbf{v}]_i \equiv \sum_{j=1}^{3} \mathcal{R}_{i,j}[v_j], \ i = 1, 2, 3.$$

(vii)

■ HÖRMANDER-MIKHLIN THEOREM:

Theorem 9 *Consider an operator $\mathcal{L}$ defined by means of a Fourier multiplier $m = m(\xi)$,*

$$\mathcal{L}[v](x) = \mathcal{F}^{-1}_{\xi\to x}\left[m(\xi)\mathcal{F}_{x\to\xi}[v](\xi)\right],$$

where $m \in L^\infty(\mathbb{R}^N)$ has classical derivatives up to order $[N/2]+1$ in $\mathbb{R}^N \setminus \{0\}$ and satisfies

$$|\partial^\alpha_\xi m(\xi)| \le c_\alpha |\xi|^{-|\alpha|}, \ \xi \neq 0,$$

for any multiindex α such that $|\alpha| \le [N/2]+1$, where $[\,]$ denotes the integer part.
Then $\mathcal{L}$ is a bounded linear operator on $L^p(\mathbb{R}^N)$ for any $1 < p < \infty$.

See Stein [251, Chap. 4, Theorem 3]. □

6 Weak Convergence of Integrable Functions

Let X be a Banach space, B_X the (closed) unit ball in X, and B_{X^*} the (closed) unit ball in the dual space X^*.

(i) Here are some known facts concerning *weak compactness*:

(1) B_X is weakly compact only if X is reflexive. This is stated in Kakutani's theorem; see Theorem III.6 in Brezis [41].
(2) B_{X^*} is weakly-(*) compact. This is stated in Banach-Alaoglu-Bourbaki theorem; see Theorem III.15 in Brezis [41].
(3) If X is separable, then B_{X^*} is sequentially weakly-(*) compact; see Theorem III.25 in Brezis [41].
(4) A non-empty subset of a Banach space X is weakly relatively compact only if it is sequentially weakly relatively compact. This is stated in Eberlein-Shmuliyan-Grothendieck theorem; see Kothe [174], Paragraph 24, 3.(8); 7.

(ii) In view of these facts:

(1) Any bounded sequence in $L^p(Q)$, where $1 < p < \infty$ and $Q \subset \mathbb{R}^N$ is a domain, is weakly (relatively) compact.
(2) Any bounded sequence in $L^\infty(Q)$, where $Q \subset \mathbb{R}^N$ is a domain, is weakly-(*) (relatively) compact.

(iii) Since L^1 is neither reflexive nor dual of a Banach space, the uniformly bounded sequences in L^1 are in general not weakly relatively compact in L^1. On the other hand, the property of weak compactness is equivalent to the property of sequential weak compactness.

■ WEAK COMPACTNESS IN THE SPACE L^1:

Theorem 10 *Let $\mathcal{V} \subset L^1(Q)$, where $Q \subset \mathbb{R}^M$ is a bounded measurable set.*
Then the following statements are equivalent:

(i) Any sequence $\{v_n\}_{n=1}^\infty \subset \mathcal{V}$ contains a subsequence weakly converging in $L^1(Q)$.
(ii) For any $\varepsilon > 0$, there exists $k > 0$ such that

$$\int_{\{|v| \geq k\}} |v(y)| \, \mathrm{d}y \leq \varepsilon \text{ for all } v \in \mathcal{V}.$$

(iii) For any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $v \in \mathcal{V}$

$$\int_M |v(y)| \mathrm{d}y < \varepsilon$$

for any measurable set $M \subset Q$ such that

$$|M| < \delta.$$

(iv) There exists a non-negative function $\Phi \in C([0, \infty))$,

$$\lim_{z \to \infty} \frac{\Phi(z)}{z} = \infty,$$

such that

$$\sup_{v \in \mathcal{V}} \int_Q \Phi(|v(y)|) \, \mathrm{d}y \leq c.$$

See Ekeland and Temam [92, Chap. 8, Theorem 1.3] and Pedregal [231, Lemma 6.4]. □

Condition (iii) is termed *equi-integrability* of a given set of integrable functions, and the equivalence of (i) is the *Dunford-Pettis theorem* (cf., e.g., Diestel [82, p.93]. Condition (iv) is called de la Vallé-Poussin criterion; see Pedregal [231, Lemma 6.4]. The statement "there exists a non-negative function $\Phi \in C([0,\infty))$" in condition (iv) can be replaced by "there exists a non-negative convex function on $[0,\infty)$."

7 Non-negative Borel Measures

(i) The symbol $C_c(Q)$ denotes the space of continuous functions with compact support in a locally compact Hausdorff metric space Q.
(ii) The symbol $\mathcal{M}(Q)$ stands for the space of signed Borel measures on Q. The symbol $\mathcal{M}^+(Q)$ denotes the cone of non-negative Borel measures on Q. A measure $\nu \in \mathcal{M}^+(Q)$ such that $\nu[Q] = 1$ is called *probability measure*.
(iii)

■ RIESZ REPRESENTATION THEOREM:

Theorem 11 *Let Q be a locally compact Hausdorff metric space. Let f be a non-negative linear functional defined on the space $C_c(Q)$.*

Then there exist a σ-algebra of measurable sets containing all Borel sets and a unique non-negative measure on $\mu_f \in \mathcal{M}^+(Q)$ such that

$$< f; g >= \int_Q g \, \mathrm{d}\mu_f \text{ for any } g \in C_c(Q). \tag{13}$$

Moreover, the measure μ_f enjoys the following properties:

- $\mu_f[K] < \infty$ *for any compact $K \subset Q$.*
- $$\mu_f[E] = \sup\left\{\mu_f[K] \mid K \subset E\right\}$$
  for any open set $E \subset Q$.
- $$\mu_f[V] = \inf\{\mu(E) \mid V \subset E, \ E \text{ open}\}$$
  for any Borel set V.
- *If E is μ_f measurable, $\mu_f(E) = 0$, and $A \subset E$, then A is μ_f measurable.*

See Rudin [239, Chap. 2, Theorem 2.14]. □

(iv) **Corollary 1** *Assume that* $f : C_c^\infty(Q) \to \mathbb{R}$ *is a linear and non-negative functional, where* Q *is a domain in* $\mathbb{R}^N$.

Then there exists a measure μ_f *enjoying the same properties as in Theorem 11 such that* f *is represented through (13).*

See Evans and Gariepy [97, Chap. 1.8, Corollary 1].

(v) If $Q \subset \mathbb{R}^M$ is a bounded set, the space $\mathcal{M}(\overline{Q})$ can be identified with the dual to the Banach space $C(\overline{Q})$ via (13). The space $\mathcal{M}(\overline{Q})$ is compactly embedded into the dual Sobolev space $W^{-k,p'}(Q)$ as soon as $Q \subset \mathbb{R}^M$ is a bounded Lipschitz domain and $kp > M$, $1/p + 1/p' = 1$ (see Evans [95, Chap. 1, Theorem 6]).

(vi) If μ is a probability measure on Ω and g a μ-measurable real-valued function, then we have *Jensen's inequality*

$$\Phi\left(\int_\Omega g \, \mathrm{d}\mu\right) \leq \int_\Omega \Phi(g) \, \mathrm{d}\mu \tag{14}$$

for any convex Φ defined on $\mathbb{R}$.

(vii)

■ APPROXIMATION OF MEASURES BY INTEGRABLE FUNCTION:

Theorem 12 *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain. Let* $\mu \in \mathcal{M}^+(\overline{\Omega})$ *be a non-negative Borel measure on* $\overline{\Omega}$.

Then there exists a sequence of functions $\{g_n\}_{n=1}^\infty$,

$$g_n \in L^1(\Omega), \ \|g_n\|_{L^1(\Omega)} \leq \|\mu\|_{\mathcal{M}^+(\overline{\Omega})}, \ g_n \geq 0 \textit{ a.a. in } \Omega \textit{ for any } n = 1, 2, \ldots$$

such that

$$g_n \to \mu \textit{ weakly-(*) in } \mathcal{M}(\overline{\Omega});$$

specifically

$$\int_\Omega g_n \varphi \, \mathrm{d}x \to \langle \mu; \varphi \rangle_{\mathcal{M}(\overline{\Omega}), C(\overline{\Omega})} \textit{ for any } \varphi \in C(\overline{\Omega}).$$

This is a very particular case of Olech [226, Sect. 2, property (4)].

Remark The same result holds true for any signed measure $\mu \in \mathcal{M}(\overline{\Omega})$. In this case, of course, the approximate functions g_n need not be non-negative.

8 Parametrized (Young) Measures

(i) Let $Q \subset \mathbb{R}^N$ be a domain. We say that $\Phi : Q \times \mathbb{R}^M$ is a *Caratheodory function* on $Q \times \mathbb{R}^M$ if

$$\left\{ \begin{array}{c} \text{for a. a. } x \in Q, \text{ the function } \lambda \mapsto \Phi(x, \lambda) \text{ is continuous on } \mathbb{R}^M; \\ \\ \text{for all } \lambda \in \mathbb{R}^M, \text{ the function } x \mapsto \Phi(x, \lambda) \text{ is measurable on } Q. \end{array} \right\} \tag{15}$$

We say that $\{\nu_x\}_{x \in Q}$ is a *family of parametrized measures* if ν_x is a probability measure for a.a. $x \in Q$ and if

$$\left\{ \begin{array}{c} \text{the function } x \to \int_{\mathbb{R}^M} \phi(\lambda)\, \mathrm{d}\nu_x(\lambda) :=< \nu_x, \phi > \text{ is measurable on } Q \\ \\ \text{for all } \phi : \mathbb{R}^M \to \mathbb{R},\ \phi \in C(\mathbb{R}^M) \cap L^\infty(\mathbb{R}^M). \end{array} \right\} \tag{16}$$

(ii)

■ FUNDAMENTAL THEOREM OF THE THEORY OF PARAMETERIZED (YOUNG) MEASURES:

Theorem 13 *Let $\{\mathbf{v}_n\}_{n=1}^\infty$, $\mathbf{v}_n : Q \subset \mathbb{R}^N \to \mathbb{R}^M$ be a weakly convergent sequence of functions in $L^1(Q; \mathbb{R}^M)$, where Q is a domain in $\mathbb{R}^N$.*

Then there exist a subsequence (not relabeled) and a parametrized family $\{\nu_y\}_{y \in Q}$ of probability measures on $\mathbb{R}^M$ depending measurably on $y \in Q$ with the following property:

For any Caratheodory function $\Phi = \Phi(y, z)$, $y \in Q$, $z \in \mathbb{R}^M$ such that

$$\Phi(\cdot, \mathbf{v}_n) \to \overline{\Phi} \text{ weakly in } L^1(Q),$$

we have

$$\overline{\Phi}(y) = \int_{\mathbb{R}^M} \Phi(y, z)\, \mathrm{d}\nu_y(z) \text{ for a.a. } y \in Q.$$

See Pedregal [231, Chap. 6, Theorem 6.2]. □

(iii) The family of measures $\{\nu_y\}_{y \in Q}$ associated to a sequence $\{\mathbf{v}_n\}_{n=1}^\infty$,

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^1(Q; \mathbb{R}^M),$$

is termed *Young measure*. We shall systematically denote by the symbol $\overline{\Phi(\cdot, \mathbf{v})}$ the weak limit associated to $\{\Phi(\cdot, \mathbf{v}_n)\}_{n=1}^\infty$ via the corresponding Young measure constructed in Theorem 13. Note that Young measure need not be unique for a given sequence.

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