

# Sequential Experiments with Primes

Mihai Caragiu

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# Preface

## Why This Book?

This book is actually about the mathematical life and destiny of mathematics faculty and their talented students in small undergraduate colleges (not necessarily the elite ones, which are different) who wish to obtain a glimpse into the ethereal world of “higher mathematics.” How can this be done, in spite of daily pressures such as high teaching and service loads for faculty and the heterogeneous and career-oriented curricular schedules for students? How can these students learn to see the value of higher mathematics? These things need to be figured out for an education that will prepare students for lifetime of learning.

My experience as a mathematics teacher at Ohio Northern University has led me to an answer that I would like to share by means of this book with other faculty members and talented students at small undergraduate colleges: it is about experimenting with elementary number theory (prime numbers and related functions) and witnessing the amazing behavior of special integer sequences. Through elementary means we managed in six years to get from scribbling a recurrence formula on a piece of paper to being spoken of at a 2012 international conference on Fibonacci numbers. For a small undergraduate college, that was a big deal.

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Mihai Caragiu

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