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Numerical Methods and Analysis of Multiscale Problems

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Preface

Multiscale problems are omnipresent in real-world applications and pose a challenge in terms of numerical approximations. Well-known examples include modeling of plates and shells, composites, neuronal modeling, and flows in porous media. Physically, what characterize such problems is the presence of different important physical scales. Mathematically, it means that derivatives of the solution might blow up as some parameter goes to zero, as in singular perturbed problems, for instance.

Accordingly, the partial differential equations (PDEs) that model these problems are characterized by either the presence of a small parameter in the equation (e.g., the viscosity of a turbulent flow) or in the domain itself (as in shell or neuroscience problems). Denote this small parameter by ε. Frequently, there is interest in investigating or approximating the solutions for positive but nonvanishing ε. However, they might behave as in the $\varepsilon = 0$ case or, even more interestingly, as in an intermediate, asymptotic state, which depends on the problem under consideration. It hardly comes as a surprise that, in general, standard numerical methods do not perform well under all regimes.

It is then important to discuss modeling of multiscale PDEs, where modeling has two meanings. It can be in the sense of approximating the original PDE by other equations that are easier to solve, as in plates or domains with rough boundaries. It can also be in numerical approximation point of view, where the final goal is to develop a numerical scheme that is robust, i.e., that works well for a wide range of parameters.

Understanding such asymptotic behaviors for different problems, and designing and analyzing robust models and numerical schemes, is the goal of these notes. As much as possible, the problems are considered in simple settings, so that technical details do not hinder the understanding of what is essential in each case considered.

The techniques involved are introduced by means of case studies, and I derive modeling error estimates by means of asymptotic analysis. The problems I describe involve equations with reaction, advection, and diffusion terms and problems with oscillatory coefficients or posed in domain with rough boundaries (like in a golf ball). I also introduce, in a general setting, some finite element methods that are suitable to deal with multiscale problems.

Prerequisites and Contents

This text is mainly oriented toward advanced undergraduate and graduate students. It might be also useful to researchers that are willing to read a bit about multiscale problem without digging too deep into technical details or sophisticated problems, as in research papers. I assume basic knowledge of analysis, mainly regarding Sobolev spaces, and some topics on functional analysis. It is however not essential that the reader completely master these results to follow the present text, and proper references are mentioned when needed. It would also be useful to have some experience with finite element methods, but the main tools are developed in these notes.

Chapter 1 introduces some basic notation and results that are useful throughout the book. It also contains a general description of several important finite element methods. Next, in Chap. 2, I consider one-dimensional singular perturbed reaction–advection–diffusion problems, in terms of numerical discretization and asymptotics. In Chap. 3, I discuss a one-dimensional equation of interest in computational neuroscience. It is a diffusion–reaction problem, with Dirac deltas (modeling the presence of synapses in a neuron) introducing layers in the interior of the domain. I show how a multiscale method can tackle the problem. A two-dimensional reaction–diffusion problem is considered in Chap. 4, and there I discuss how to develop the boundary layer terms for two-dimensional domains and introduce a multiscale finite element method based on enriching finite element spaces. Chapter 5 concerns PDEs posed in domains with rough boundaries, and I develop asymptotic expansions of the solutions and propose and analyze a finite element method of multiscale type. Finally, Chap. 6 deals with the classical problem of elliptic PDEs with rough coefficient. I develop its asymptotic expansion and analyze a numerical method in a simple one-dimensional setting.

Acknowledgments

Of course this book would never be possible without many enlightening iterations with wonderful researchers. I was introduced to the field of numerical analysis of multiscale problems by the late Leopoldo Franca and then proceeded to work under the guidance of Doug Arnold. Both of them fundamentally shaped my career. The influence of my longtime friend and main collaborator Frédéric Valentin is of order ε^{-1}, and the choice of topics in the book reflects that. He also read the manuscript and offered suggestions that improved the notes.

Parts of this book were drawn from [184, Chap. 4] and short courses delivered at Denver University, USA; Universitá degli Studi di Pavia, Italy; and Brazilian Universities, and I thank Leo Franca, Daniele Boffi, Carlo Lovadina, Paulo Bösing, Igor Mozolevski, and Sandra Malta for the kind invitations. Several other people influenced in manifold ways the outcome of the book. I am particularly grateful to

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This book was completed while I was spending a sabbatical year at the Division of Applied Mathematics at Brown University. During such period, I was lucky enough to learn from my host Johnny Gúzman and also Marcus Sarkis from WPI. Their wit and hospitality and the general atmosphere of the Division confirmed that top-notch mathematics and friendliness are definitely compatible.

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I dedicate this work to my wife Daniele and daughter Maria. They make life wonderful.

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