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Alexander John Taylor

Analysis of Quantised Vortex Tangle

Doctoral Thesis accepted by
the University of Bristol, Bristol, England

Springer

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Tanto monta cortar como desatar
(Cutting is as good as untying)
Topologically misguided Spanish proverb

Supervisor's Foreword

I am pleased that Springer is publishing Alexander Taylor's exceptional thesis. The quantised vortex tangle he analyzes is found in random waves: mathematical models of interference in three-dimensional space displaying great complexity, which are very different from the simple structure of textbook plane waves. The most significant features of these waves are vortices, which are stationary filaments in the interference pattern, and zeros in the wave's intensity around which the energy flows. As centres of rotation of a surrounding flow (such as in a fluid), vortices hold a fascination for physicists. Vortices in waves are also phase singularities, meaning that the wave field's phase (represented by the argument of the wave's complex amplitude) is not defined on a vortex, and the full cycle of phases occurs on any loop around it. A wave field's topology is thus manifested in its vortex structure. This thesis describes a further topology—the kind of knotting exhibited in the tangle of vortex lines, which is explored using computer models of wave interference in several distinct model systems.

Although vortices were understood as a general phenomenon in wave interference over 40 years ago (by John Nye and Michael Berry, in Bristol), their study has intensified in the last couple of decades, especially in light waves, where the experimental manipulation of laser beams is enabled via holograms, which has led to the fields of singular optics and structured light. Indeed, laser beams can be be structured to contain knotted and linked vortices (in theoretical work by Michael Berry, Robert King and myself, and realised experimentally by the group of Miles Padgett in Glasgow).

On the other hand, random waves are a universal and natural feature in many kinds of wave, such as the surface of the sea, in speckle patterns of light reflected from rough surfaces, in acoustic noise in a room, or quantum matter waves in a chaotic billiard cavity. The natural mathematical model for such waves is simply a sum of plane waves, with random directions, amplitudes, and phases, whose structure is surprisingly rich and subtle. Michael Berry and I, in our calculation of the local statistical properties of vortices in this model (including their density and curvatures), speculated that systems of random waves could contain knotted vortices, although there are no known techniques to answer that question analytically. Subsequent numerical experiments on optical speckle, with Kevin O'Holleran and Miles Padgett in Glasgow, found linked vortex loops to be relatively common, but no examples of knots.

Alexander's thesis addresses this problem by large scale computation, generating random wave fields in volumes much larger than previously: vortex filaments are tracked in the 3D volume, and then calculations are made of topological knot invariants for vortex loops, and fractal properties of the entire tangle. The vortex tracking method is a refinement of previous methods used to follow vortex lines, which resolves the volume of the interference pattern into voxels, and candidate vortices as phase singularities are located in edges of the voxel lattice. In a generalization of the classical bisection method for finding roots, Alexander's procedure re-resolves these regions of the volume, significantly improving the resolution of the vortex pattern. With these improvements, quantitative agreement is possible with the analytically-derived, smooth curvature statistics of random vortices, and the methods were successfully extended to find distributions of other interesting geometric quantities such as persistence length and torsion.

The topological identification algorithm also improves previously published methods: rather than relying on a single discriminating knot invariant, which may be computationally costly to calculate for a complicated, tangled loop, instead several simpler invariants are calculated, whose fingerprint has sufficient discriminatory power for unambiguous identification of most tabulated knots to a reasonable level of complexity (up to around nine crossings). Not only are knots found to occur in random waves, but the statistics of many thousands of random wave simulations in several systems show a diverse range of different knots with varying topological complexity, which is different in the three principal systems (plane waves in a cube with periodic boundary conditions, hyperspherical harmonics in the 3-sphere, and eigenfunctions of the 3D quantum harmonic oscillator). The main results were extended and published after Alexander's Thesis was completed, in Taylor and Dennis, Nature Communications vol 7, article 12346 (2016).

The results are of interest to students and scientists working in singular optics, quantum chaos and quantum turbulence, where random wave models and vortex tangles occur. Mathematicians interested in random geometry and topology may find grounds for new conjectures in statistical topology, an area where exact, analytical results are notoriously rare; only a little is known about the topology of these systems at lengthscales beyond a few wavelengths. Knotted random walks, which random tangled vortices exemplify, are very important in polymer physics, and there are tantalizing connections in the kinds of knotting topology exhibited in different systems which are explored in the thesis.

Alexander's work fits into the rich tradition of singular optics and wave physics from Bristol Theoretical Physics going back over four decades. The results described here are only possible with modern high-performance computers, which are necessary for the recognition of highly complex knots. Alexander has made this subject his own, and, as with the best graduate students, he has changed the way I, as his supervisor, approach the subject. I am especially pleased that Alexander's work answers many questions posed in my own PhD thesis. His thesis provides an excellent summary of the results and provides a background to this intriguing area. We are very grateful to the EPSRC and the Leverhulme Trust for financial support which has made this work possible.

Bristol, UK
September 2016

Prof. Mark Dennis

Abstract

This thesis is an investigation of the tangled vortex lines that arise in the interference of complex waves in three dimensions; they are nodal lines of the intensity where both the real and imaginary components of the wave field cancel out, and are singularities of the complex phase about which it sweeps out a quantised total change. We investigate the behaviour of this tangle as expressed in random degenerate eigenfunctions of the 3-torus, 3-sphere and quantum harmonic oscillator as models for wave chaos, in which many randomly weighted interfering waves produce a statistically characteristic vortex ensemble.

The geometrical and topological nature of these vortex tangles is examined via large-scale numerical simulations of random wave fields; local geometry is recovered with sufficient precision to confirm the connection to analytical random wave models, but we also recover the (high order) torsion that appears analytically inaccessible, and quantify the different length scales along which vortex lines decorrelate. From our simulations we recover statistics also on much larger scales, confirming a fractality of individual vortices consistent with random walks but also comparing and contrasting the scaling of the full vortex ensemble with other models of filamentary tangle.

The nature of the tangling itself is also investigated, geometrically where possible but in particular topologically by testing directly whether vortex curves are knotted or linked with one another. We confirm that knots and links exist, but find their statistics greatly influenced by the nature of the random wave ensemble; vortices in the 3-torus are knotted far less than might be expected from their scales of geometrical decorrelation, but in the 3-sphere and harmonic oscillator exhibit more common and more complex topology. We discuss how this result relates to the construction of each system, and finish with brief discussion of some selected topological observations.

Acknowledgements

I have to start by thanking my supervisor, Mark Dennis. I cannot imagine what Ph.D. life would have been like without his remarkable enthusiasm and dedication, and not least for suggesting such an interesting project in the first place!

I also owe much to the discussions with academics at both Bristol and elsewhere. In particular, thanks to Rami Band for his help and discussions academic and otherwise, to Stu Whittington for allowing me to probe his fount of knowledge and the illuminating insights that always followed, and in Bristol to James Annett, Michael Berry and John Hannay for the useful and interesting suggestions and interactions throughout the last few years. But these are just those whose paths I have crossed the most; thanks to all of the Bristol Physics Theory Group, including those no longer around, for contributing to such a welcoming environment with such diverse goings on.

It is important that I not forget the roles of my fellow students, particularly those I have shared an office with at one time or another. Dare I say, without you it would have been really boring! Oh, and to James Ring, I just noticed your own acknowledgement of me; suffice to say I was as delighted to prattle on as you apparently were to listen.

Outside the office, thanks to everyone with whom I have played Go, juggled, shared a flat (that's you Richard!) or even just a drink, for I am not sustained by science alone. Special mention to the Kivy team, they are pretty cool dudes. And a little outside the box, I am grateful to everyone that contributed to the scientific software ecosystem from which I've drawn heavily—when it comes down to it, this would have been way harder without that stuff.

To my parents, it is cliché to say it but yes; I have finished writing now. I could not have done it without you, so thanks for everything! (actually really now).

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