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Finite Element Methods for Incompressible Flow Problems

 Springer

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For Anja and Josephine

Preface

Incompressible flow problems appear in many models of physical processes and applications. Their numerical simulation requires in particular a spatial discretization. Finite element methods belong to the mathematically best understood discretization techniques.

This monograph is devoted mainly to the mathematical aspects of finite element methods for incompressible flow problems. It addresses researchers, Ph.D. students, and even students aiming for the master's degree. The presentation of the material, in particular of the mathematical arguments, is performed in detail. This style was chosen in the hope to facilitate the understanding of the topic, especially for nonexperienced readers.

Most parts of this monograph were presented in three consecutive master's level courses taught at the Free University of Berlin, and this monograph is based on the corresponding lecture notes. First of all, I like to thank the students who attended these courses. Many of them wrote finally their master's thesis under my supervision. Then, I like to thank two collaborators of mine, Julia Novo (Madrid) and Gabriel R. Barrenechea (Glasgow), who read parts of this monograph and gave valuable suggestions for improvement. Above all, I like to thank my beloved wife Anja and my daughter Josephine for their continual encouragement. Their efforts to manage our daily life and to save me working time were an invaluable contribution for writing this monograph in the past 3 years.

Colbitz, Germany
July 2016

Volker John

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