

Classical Mechanics

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Classical Mechanics

Hamiltonian and Lagrangian Formalism

Second Edition

 Springer

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Preface to the Second Edition

In the revised and enlarged edition I corrected numerous errors and typos found in the first edition. In the Chaps. 1, 2, 7 and 8 I have made, in the interests of clarity, a large number of small changes. Some sections, which even I myself was not able to understand 5 years after they were written, were rewritten again. Several new sections with applications of formalism were added: 1.7.7, 6.9, 8.7.1 and 8.7.2. The main alteration in this edition consists of the incorporation of a new chapter devoted to the description of a relativistic spinning particle in external fields. It can be considered as a non-trivial application of the formalism of constrained systems described in Chap. 8. A review of the achievements in this fascinating area before 1968 can be found in the book of Corben [58]. Contrary to Corben, who discussed the problem on the level of equations of motion, my emphasis has been placed on the Lagrangian and Hamiltonian variational formulations for the description of rotational degrees of freedom and their influence on the trajectory of a spinning body. I present the so-called vector model of spin and show that it provides a unified conceptual framework, allowing to collect and tie together a lot of ideas and achievements accumulated on the subject after publication of the Corben's book.

Juiz de Fora, MG, Brazil
August 2016

Alexei Deriglazov

Preface to the First Edition

Formalism of classical mechanics underlies a number of powerful mathematical methods, widely used in theoretical and mathematical physics [1–11]. In these lectures we present some selected topics of classical mechanics, which may be useful for graduate-level students intending to work in one of the branches of a vast field of theoretical physics. Except for the last chapter, which is devoted to the discussion of singular theories and their local symmetries, the topics selected correspond to the standard course of classical mechanics.

For the convenience of the reader, we have tried to make the material of different chapters as independent as possible. So, the reader who is familiar with Lagrangian mechanics can proceed to any one of Chaps. 3–8 after reading the second chapter.

In our presentation of the material we have tried, where possible, to replace intuitive motivations and “scientific folklore” by exact proofs or direct computations. To illustrate how classical-mechanics formalism works in other branches of theoretical physics, we have presented examples related to electrodynamics as well as to relativistic and quantum mechanics. Most of the suggested exercises are directly related to the main body of the text.

While in some cases the formalism is developed beyond the traditional level adopted in the standard textbooks on classical mechanics [12–14], the only mathematical prerequisites are some knowledge of calculus and linear algebra.

In the frameworks of classical and quantum theories, the Hamiltonian and Lagrangian formulations each have advantages and disadvantages. Since our focus here is Hamiltonian mechanics, let us mention some of the arguments for using this type of formalism.

- There is a remarkable democracy between variables of position and velocity in Nature: being *independent* one from another, they contain complete information on the properties of a classical system at a given instance. Besides, just the positions and velocities at the initial instant of time are necessary and sufficient to predict an evolution of the system. In Lagrangian formalism this democracy, while reflected in the initial conditions, is not manifest in the course of evolution, since only variables of position are treated as independent in Lagrangian

equations. Hamiltonian formalism restores this democracy, treating positions and velocities on equal footing, as independent coordinates that parameterize a phase space.

- According to the canonical quantization paradigm, the construction of the Hamiltonian formulation for a given classical system is the first necessary step in the passage from classical to quantum theory. It is sufficient to point out that quantum evolution in the Heisenberg picture is obtained from the Hamiltonian equations through replacement of the phase-space variables by corresponding operators. As to the operators, their commutators are required to resemble the Poisson brackets of the phase-space variables.
- The conventional way to describe a relativistic theory is to formulate it in terms of a singular Lagrangian (the singularity is the price we pay for the manifest relativistic invariance of the formulation). It implies a rather complicated structure of Lagrangian equations, which may consist of both second- and first-order differential equations as well as algebraic ones. Besides, there may be identities present among the equations, which implies functional arbitrariness in the corresponding solutions. It should be mentioned that, in the modern formulation, physically interesting theories (electrodynamics, gauge field theories, the standard model, string theory, etc.) are of this type. In this case, Hamiltonian formulation gives a somewhat clearer geometric picture of classical dynamics [8]: all the solutions are restricted to lying on some surface in the phase space, while the abovementioned arbitrariness is avoided by postulating classes of equivalent trajectories. Physical quantities are then represented by functions defined in these classes. The procedure for investigation of this picture is based entirely on the use of special coordinates adopted to the surface, which in turn require a rather detailed development of the theory of canonical transformations. Altogether Hamiltonian formulation leads to a self-consistent physical interpretation of a general singular theory, forming the basis for numerous particular prescriptions and approaches to quantization of concrete theories [10].

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Alexei Deriglazov

Contents

1	Sketch of Lagrangian Formalism	1
1.1	Newton's Equation	1
1.2	Galilean Transformations: Principle of Galilean Relativity	12
1.3	Poincaré and Lorentz Transformations: The Principle of Special Relativity	18
1.4	Principle of Least Action	29
1.4.1	Variational Analysis.....	31
1.4.2	Generalized Coordinates, Coordinate Transformations and Symmetries of an Action.....	36
1.5	Examples of Continuous (Field) Systems	46
1.6	Action of a Constrained System	53
1.6.1	The Recipe	54
1.6.2	Justification of the Recipe	60
1.6.3	Description of Constrained System by Singular Action ...	62
1.6.4	Kinetic Versus Potential Energy: Forceless Mechanics of Hertz	64
1.7	Electromagnetic Field in Lagrangian Formalism	66
1.7.1	Maxwell Equations	66
1.7.2	Nonsingular Lagrangian Action of Electrodynamics.....	69
1.7.3	Manifestly Poincaré-Invariant Formulation in Terms of a Singular Lagrangian Action.....	74
1.7.4	Notion of Local (Gauge) Symmetry	76
1.7.5	Lorentz Transformations of Three-Dimensional Potential: Role of Gauge Symmetry	79
1.7.6	Relativistic Particle in Electromagnetic Field	80
1.7.7	Speed of Light and Critical Speed in External Field	83
1.7.8	Poincaré Transformations of Electric and Magnetic Fields	86

2	Hamiltonian Formalism	91
2.1	Derivation of Hamiltonian Equations	91
2.1.1	Preliminaries	91
2.1.2	From Lagrangian to Hamiltonian Equations.....	95
2.1.3	Short Prescription for Hamiltonization Procedure, Physical Interpretation of Hamiltonian	99
2.1.4	Inverse Problem: From Hamiltonian to Lagrangian Formulation	101
2.2	Poisson Bracket and Symplectic Matrix	101
2.3	General Solution to Hamiltonian Equations	104
2.4	Picture of Motion in Phase Space	108
2.5	Conserved Quantities and the Poisson Bracket	110
2.6	Phase Space Transformations and Hamiltonian Equations	113
2.7	Definition of Canonical Transformation	116
2.8	Generalized Hamiltonian Equations: Example of Non-canonical Poisson Bracket	117
2.9	Hamiltonian Action Functional	121
2.9.1	Schrödinger Equation as the Hamiltonian System	122
2.9.2	Lagrangian Action Associated with the Schrödinger Equation. Analogies Between Quantum Mechanics and Electrodynamics	123
2.9.3	Probability as a Conserved Charge via the Noether Theorem	127
2.9.4	First-Order Action Functional, Routhian and All That	129
2.10	Hamiltonization of a Theory with Higher-Order Derivatives.....	130
2.10.1	First-Order Trick	130
2.10.2	Ostrogradsky Method	132
3	Canonical Transformations of Two-Dimensional Phase Space	137
3.1	Time-Independent Canonical Transformations.....	137
3.1.1	Time-Independent Canonical Transformations and Symplectic Matrix.....	137
3.1.2	Generating Function	139
3.2	Time-Dependent Canonical Transformations	141
3.2.1	Canonical Transformations and Symplectic Matrix	142
3.2.2	Generating Function	144
4	Properties of Canonical Transformations	147
4.1	Invariance of the Poisson Bracket (Symplectic Matrix)	148
4.2	Infinitesimal Canonical Transformations: Hamiltonian as a Generator of Evolution	153
4.2.1	Generator of Infinitesimal Canonical Transformation.....	154

- 4.3 Generating Function of Canonical Transformation 156
 - 4.3.1 Free Canonical Transformation and Its
Function $F(q', p', \tau)$ 157
 - 4.3.2 Generating Function $S(q, q', \tau)$ 158
- 4.4 Examples of Canonical Transformations 161
 - 4.4.1 Evolution as a Canonical Transformation:
Invariance of Phase-Space Volume 161
 - 4.4.2 Canonical Transformations in Perturbation Theory 164
 - 4.4.3 Coordinates Adjusted to a Surface 165
- 4.5 Transformation Properties of the Hamiltonian Action 166
- 4.6 Summary: Equivalent Definitions for Canonical Transformation ... 167
- 4.7 Hamilton–Jacobi Equation 168
- 4.8 Action Functional as a Generating Function of Evolution 172
- 5 Integral Invariants** 175
 - 5.1 Poincaré–Cartan Integral Invariant 175
 - 5.1.1 Preliminaries 175
 - 5.1.2 Line Integral of a Vector Field, Hamiltonian
Action, Poincaré–Cartan and Poincaré Integral
Invariants 177
 - 5.1.3 Invariance of the Poincaré–Cartan Integral 180
 - 5.2 Universal Integral Invariant of Poincaré 183
- 6 Some Mechanical Problems in a Geometric Setting** 189
 - 6.1 Analysis of Trajectories and the Principle of Maupertuis 189
 - 6.1.1 Trajectory: Separation of Kinematics from Dynamics 190
 - 6.1.2 Equations for Trajectory in the Hamiltonian
Formulation 192
 - 6.1.3 The Principle of Maupertuis for Trajectories 194
 - 6.1.4 Lagrangian Action for Trajectories 194
 - 6.2 Description of a Potential Motion in Terms of a Pair
of Riemann Spaces 197
 - 6.3 Basic Notions of Riemann Geometry 200
 - 6.3.1 Riemann Space 200
 - 6.3.2 Covariant Derivative and Riemann Connection 206
 - 6.3.3 Parallel Transport: Notions of Covariance
and Coordinate Independence 208
 - 6.4 Definition of Covariant Derivative Through Parallel
Transport: Formal Solution to the Parallel Transport Equation 213
 - 6.5 The Geodesic Line and Its Reparametrization
Covariant Equation 214
 - 6.5.1 Reparametrization Covariant Equation of the
Geodesic Line 214
 - 6.6 Example: A Surface Embedded in Euclidean Space 217
 - 6.7 Shortest Line and Geodesic Line: One More Example
of a Singular Action 219

6.8	Formal Geometrization of Mechanics	223
6.9	Three-Dimensional Acceleration and Speed of Light in General Relativity	225
7	Transformations, Symmetries and Noether Theorem	235
7.1	The Notion of Invariant Action Functional	236
7.2	Coordinate Transformation, Induced Transformation of Dynamical Variables and Symmetries of an Action	240
7.3	Examples of Invariant Actions, Galileo Group	243
7.4	Poincaré Group, Relativistic Particle	246
7.5	Symmetries of Equations of Motion	248
7.6	Noether Theorem	250
7.7	Infinitesimal Symmetries	253
7.8	Discussion of the Noether Theorem	256
7.9	Use of Noether Charges for Reduction of the Order of Equations of Motion	257
7.10	Examples	258
7.11	Reparametrization Invariance as a Local Symmetry	261
7.12	Symmetries of Hamiltonian Action	262
	7.12.1 Infinitesimal Symmetries Given by Canonical Transformation and by Charge	262
	7.12.2 Structure of Infinitesimal Symmetry of a General Form	264
	7.12.3 Hamiltonian Versus Lagrangian Global Symmetry	268
8	Hamiltonian Formalism for Singular Theories	271
8.1	Hamiltonization of a Singular Theory: The Recipe	272
	8.1.1 Two Toy Models	272
	8.1.2 Dirac Procedure	277
8.2	Justification of the Hamiltonization Recipe	282
	8.2.1 Configuration-Velocity Space	282
	8.2.2 Hamiltonization	284
	8.2.3 Comparison with the Dirac Recipe	287
8.3	Classification of Constraints	289
8.4	Classical Observables and Physical Content of a Degenerate Theory	291
8.5	Theory with Second-Class Constraints: Dirac Bracket	296
8.6	Examples of Theories with Second-Class Constraints	301
	8.6.1 Mechanics with Kinematic Constraints	301
	8.6.2 Singular Lagrangian Action Underlying the Schrödinger Equation	303
8.7	Examples of Theories with First-Class Constraints	305
	8.7.1 Classical Mechanics in Reparametrization- Invariant Form and the Schrödinger Equation	305
	8.7.2 Relativistic Particle: Three Basic Formulations	308
	8.7.3 Electrodynamics	319

8.8	Local Symmetries and Constraints.....	321
8.8.1	Symmetries of Lagrangian and Hamiltonian Formulations	322
8.8.2	Local Symmetry Does Not Imply a Conserved Charge	329
8.8.3	Formalism of Extended Lagrangian.....	329
8.8.4	Local Symmetries of the Extended Lagrangian: Dirac Conjecture	334
8.8.5	Local Symmetries of the Initial Lagrangian	338
8.8.6	Conversion of Second-Class Constraints by Deformation of Lagrangian Local Symmetries	342
8.8.7	Conversion in Maxwell–Proca Lagrangian for Massive Vector Field	349
9	Classical and Quantum Relativistic Mechanics of a Spinning Particle	353
9.1	Vector Model of Spinning Particle: Non Relativistic Spin.....	354
9.1.1	Description of Non Relativistic Spin on the Base of Two Constraints	358
9.1.2	An Example of Classical Mechanics Without Observable Trajectories	363
9.1.3	Description of Non Relativistic Spin on the Base of Three Constraints.....	364
9.2	Canonical Quantization and Pauli Equation	366
9.3	The Strange Quantum Mechanics of Dirac Equation, or Why We Need a Semiclassical Model of Relativistic Spin?.....	369
9.4	Spin-Tensor of Frenkel and Lorentz Covariant Form of Spin Fiber Bundle \mathbb{T}^4	372
9.5	Four-Dimensional Spin-Vector, Pauli-Lubanski Vector and Bargmann-Michel-Telegdi Vector	376
9.6	Search for the Lagrangian of Relativistic Spinning Particle	377
9.6.1	Variational Problem for Prescribed Dirac’s Constraints....	377
9.6.2	Interaction and the Problem of Covariant Formalism	379
9.6.3	Particle with the Fundamental Length Scale.....	381
9.6.4	Classification of Vector Models	382
9.7	Interaction with Electromagnetic Field	383
9.7.1	Manifestly Covariant Hamiltonian Formulation.....	384
9.7.2	Lagrangian Equations of Motion and Comparison with Approximate Equations of Frenkel and Bargmann-Michel-Telegdi	388
9.7.3	Parametrization of Physical Time and Physical Hamiltonian	392
9.8	First Relativistic Corrections and Fine Structure of Hydrogen Spectrum	395
9.9	Fast Spinning Particle and Rainbow Geometry	397
9.10	Interaction with Gravitational Field	401

9.10.1	Lagrangian of Spinning Particle with Gravimagnetic Moment	402
9.10.2	Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) Equations of a Rotating Body and Spinning Particle without Gravimagnetic Moment	408
9.10.3	Ultra-Relativistic Limit: The Problems with MPTD-Equations	411
9.10.4	Rotating Body with Gravimagnetic Moment $\kappa=1$	416
9.11	One-Particle Quantum Mechanics of a Spinning Particle, Canonical Formalism	418
9.12	Relativistic Covariance of Canonical Formalism	424
9.12.1	Relativistic Quantum Mechanics of Two-Component Klein-Gordon Equation	425
9.12.2	Covariant Operators of Vector Model	428
9.12.3	Proof of Relativistic Covariance	429
9.12.4	Relation with Dirac Equation.....	433
Bibliography		437
Index		441

Notation and Conventions

The terminology of classical mechanics is not universal. To avoid any confusion, the quantities of the configuration (phase) space are conventionally called Lagrangian (Hamiltonian) quantities.

Generalized coordinates of the configuration space are denoted by q^a . Latin indices from the beginning of the alphabet a, b, c , and so on generally range from 1 to n , $a = 1, 2, \dots, n$.

Phase-space coordinates are often denoted by one letter $Z^i = (q^a, p_b)$. Latin indices from the middle of the alphabet i, j, k , and so on generally range from 1 to $2n$, $i = 1, 2, \dots, 2n$.

Greek indices from the beginning of the alphabet α, β, γ are used to denote some subgroup of the group of variables, for example $q^\alpha = (q^1, q^\alpha)$, $\alpha = 2, 3, \dots, n$.

Repeated indices are generally summed, unless otherwise indicated. The “up” and “down” position of the index of any quantity is fixed. For example, we write q^a , p_b and never any other way.

Time variable is denoted either by τ or by t . A dot over any quantity denotes the time derivative of that quantity

$$\dot{q}^a = \frac{dq^a}{d\tau},$$

while partial derivatives are denoted by

$$\frac{\partial L(q)}{\partial q^a} = \partial_a L, \quad \frac{\partial H(Z)}{\partial Z^i} = \partial_i H.$$

The same symbol is generally used to denote a variable and a function. For example, we write $Z^i = Z^i(Z^j)$, instead of the expression $Z^i = f^i(Z^j)$ for the change of coordinates.

The notation

$$F(q, v)|_{v(z)} \equiv F(q, v)|_{v=v(z)} \equiv F(q, v),$$

implies the substitution of the function $v^a(z)$ in place of the variable v^a .

Minkowski metric is $\eta = (-1, +1, +1, +1)$.