Classical Mechanics

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Hamiltonian and Lagrangian Formalism

Second Edition

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Preface to the Second Edition

In the revised and enlarged edition I corrected numerous errors and typos found in the first edition. In the Chaps. 1, 2, 7 and 8 I have made, in the interests of clarity, a large number of small changes. Some sections, which even I myself was not able to understand 5 years after they were written, were rewritten again Several new sections with applications of formalism were added: 1.7.7, 6.9, 8.7.1 and 8.7.2. The main alteration in this edition consists of the incorporation of a new chapter devoted to the description of a relativistic spinning particle in external fields. It can be considered as a non-trivial application of the formalism of constrained systems described in Chap. 8. A review of the achievements in this fascinating area before 1968 can be found in the book of Corben [58]. Contrary to Corben, who discussed the problem on the level of equations of motion, my emphasis has been placed on the Lagrangian and Hamiltonian variational formulations for the description of rotational degrees of freedom and their influence on the trajectory of a spinning body. I present the so-called vector model of spin and show that it provides a unified conceptual framework, allowing to collect and tie together a lot of ideas and achievements accumulated on the subject after publication of the Corben's book.

Juiz de Fora, MG, Brazil — Alexei Deriglazov
August 2016

Preface to the First Edition

Formalism of classical mechanics underlies a number of powerful mathematical methods, widely used in theoretical and mathematical physics [1–11]. In these lectures we present some selected topics of classical mechanics, which may be useful for graduate-level students intending to work in one of the branches of a vast field of theoretical physics. Except for the last chapter, which is devoted to the discussion of singular theories and their local symmetries, the topics selected correspond to the standard course of classical mechanics.

For the convenience of the reader, we have tried to make the material of different chapters as independent as possible. So, the reader who is familiar with Lagrangian mechanics can proceed to any one of Chaps. 3–8 after reading the second chapter.

In our presentation of the material we have tried, where possible, to replace intuitive motivations and "scientific folklore" by exact proofs or direct computations. To illustrate how classical-mechanics formalism works in other branches of theoretical physics, we have presented examples related to electrodynamics as well as to relativistic and quantum mechanics. Most of the suggested exercises are directly related to the main body of the text.

While in some cases the formalism is developed beyond the traditional level adopted in the standard textbooks on classical mechanics [12–14], the only mathematical prerequisites are some knowledge of calculus and linear algebra.

In the frameworks of classical and quantum theories, the Hamiltonian and Lagrangian formulations each have advantages and disadvantages. Since our focus here is Hamiltonian mechanics, let us mention some of the arguments for using this type of formalism.

- There is a remarkable democracy between variables of position and velocity in Nature: being *independent* one from another, they contain complete information on the properties of a classical system at a given instance. Besides, just the positions and velocities at the initial instant of time are necessary and sufficient to predict an evolution of the system. In Lagrangian formalism this democracy, while reflected in the initial conditions, is not manifest in the course of evolution, since only variables of position are treated as independent in Lagrangian

equations. Hamiltonian formalism restores this democracy, treating positions and velocities on equal footing, as independent coordinates that parameterize a phase space.

- According to the canonical quantization paradigm, the construction of the Hamiltonian formulation for a given classical system is the first necessary step in the passage from classical to quantum theory. It is sufficient to point out that quantum evolution in the Heisenberg picture is obtained from the Hamiltonian equations through replacement of the phase-space variables by corresponding operators. As to the operators, their commutators are required to resemble the Poisson brackets of the phase-space variables.
- The conventional way to describe a relativistic theory is to formulate it in terms of a singular Lagrangian (the singularity is the price we pay for the manifest relativistic invariance of the formulation). It implies a rather complicated structure of Lagrangian equations, which may consist of both second- and first-order differential equations as well as algebraic ones. Besides, there may be identities present among the equations, which implies functional arbitrariness in the corresponding solutions. It should be mentioned that, in the modern formulation, physically interesting theories (electrodynamics, gauge field theories, the standard model, string theory, etc.) are of this type. In this case, Hamiltonian formulation gives a somewhat clearer geometric picture of classical dynamics [8]: all the solutions are restricted to lying on some surface in the phase space, while the abovementioned arbitrariness is avoided by postulating classes of equivalent trajectories. Physical quantities are then represented by functions defined in these classes. The procedure for investigation of this picture is based entirely on the use of special coordinates adopted to the surface, which in turn require a rather detailed development of the theory of canonical transformations. Altogether Hamiltonian formulation leads to a self-consistent physical interpretation of a general singular theory, forming the basis for numerous particular prescriptions and approaches to quantization of concrete theories [10].

Juiz de Fora, MG, Brazil Alexei Deriglazov
July 2010

Contents

Notation and Conventions

The terminology of classical mechanics is not universal. To avoid any confusion, the quantities of the configuration (phase) space are conventionally called Lagrangian (Hamiltonian) quantities.

Generalized coordinates of the configuration space are denoted by q^a. Latin indices from the beginning of the alphabet a, b, c, and so on generally range from 1 to n, $a = 1, 2, \ldots, n$.

Phase-space coordinates are often denoted by one letter $Z^i = (q^a, p_b)$. Latin indices from the middle of the alphabet i, j, k, and so on generally range from 1 to $2n$, $i = 1, 2, \ldots, 2n$.

Greek indices from the beginning of the alphabet α, β, γ are used to denote some subgroup of the group of variables, for example $q^a = (q^1, q^\alpha)$, $\alpha = 2, 3, \ldots, n$.

Repeated indices are generally summed, unless otherwise indicated. The "up" and "down" position of the index of any quantity is fixed. For example, we write q^a, p_b and never any other way.

Time variable is denoted either by τ or by t. A dot over any quantity denotes the time derivative of that quantity

$$\dot{q}^a = \frac{dq^a}{d\tau},$$

while partial derivatives are denoted by

$$\frac{\partial L(q)}{\partial q^a} = \partial_a L, \qquad \frac{\partial H(Z)}{\partial Z^i} = \partial_i H.$$

The same symbol is generally used to denote a variable and a function. For example, we write $Z'^i = Z'^i(Z^j)$, instead of the expression $Z'^i = f^i(Z^j)$ for the change of coordinates.

The notation

$$F(q,\upsilon)|_{\upsilon(Z)} \equiv F(q,\upsilon)|_{\upsilon=\upsilon(Z)} \equiv F(q,\upsilon)|,$$

implies the substitution of the function $\upsilon^a(z)$ in place of the variable υ^a.

Minkowski metric is $\eta = (-1,+1,+1,+1)$.