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The Early Period of the Calculus of Variations

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We dedicate this book
to our respective grandchildren
Michela, Gabriele and Andrea

Preface

> In the current state of analysis we may regard these discussions [of past mathematics] as useless, for they concern forgotten methods, which have given way to others more simple and more general. However, such discussions may yet retain some interest for those who like to follow step by step the progress of analysis, and to see how simple and general methods are born from particular questions and complicated and indirect procedures.[1]
>
> J.L. Lagrange, *Leçons sur le calcul des fonctions*, Paris 1806, p. 436.

The early history of the Calculus of variations is a well-beaten track; for instance, we refer the reader to

- The last two chapters of the *Calcul des functions* of Lagrange [152];
- *A Treatise on Isoperimetrical Problems and the Calculus of Variations* by R. Woodhouse, reprinted by Chelsea with the title *A History of the Calculus of Variations in the Eighteenth century*, [202];
- The surveys by C. Carathéodory

  (1) *The beginning of research in the Calculus of variations*, [48],
  (2) *Basel und der Begin der Variationsrechnung*, [49],
  (3) *Einführung in Eulers Arbeiten über Variationsrechnung*, [50],

  and the two volumes

  (1) *Variationsrechnung*, [46],
  (2) *Geometrische Optik*, [47];

[1] Quoted in [102], the original being

Mais dans l'état actuel de l'analyse, on peut regarder ces discussions comme inutiles, parceque elles regardent des méthodes oubliées, comme ayant fait place à d'autres plus simples et plus générales. Cependant elle peuvent avoir encore quelqu'intérêt pour ceux qui aiment suivre pas à pas les progrès de l'analyse, et à voir comment les méthodes simple et générales naissent des questions particulières et des procédés indirects et compliqués.

- the very detailed survey of the history of the one-dimensional calculus of variations from the origin until the beginning of last century by Goldstine [117], and by Thiele [193], which includes also some multidimensional calculus;
- the papers by Fraser [102, 104, 105]

Finally, we mention the two introductions to classical calculus of variations [33] and [112] which contain some historical references.

Nevertheless, we would like to go over once again presenting the most relevant continental contributions to the calculus of variations in the eighteenth century. Our main goal is to illustrate the mathematics of its founders. In doing this, we always follow very closely the original papers in their mathematical context and often in an almost literal way, adding, when we feel it is useful, our mathematical comment or complementing their proofs; however, we keep our additions separate from the original presentation. Here and there, we also comment in terms of modern mathematics. In fact, we think that this may help the reader to make clearer what ancient authors were doing, the difficulties they had to face, mistakes they made and how they were able to handle the matter following their approaches. We added the final Sect. 7.6 to make the reader, not necessarily an expert, aware of the end of the story, that is, of how the entire material is treated today.

Our book is addressed not only to historians of mathematics, but also to mathematicians who want to follow "step by step the progress of analysis" and to students of mathematics who, this way, may see the forming of a beautiful theory and the evolving of mathematical methods and techniques. This way, we hope that our work may help in getting a better understanding of the mathematical results, of the methods and techniques to obtain them, as well as of the mathematical historical context in which it all developed. Of course, in doing that, we take advantage of the wide literature that we have partly already mentioned and to which we would like to acknowledge our gratitude.

We now shortly outline the content of each chapter. We begin with an introductory chapter where, after stating Johann Bernoulli's challenge that marks the beginning of the calculus of variations, we briefly illustrate issues that belong to periods before the challenge and are especially relevant for our story: Fermat's principle of least action, which plays a crucial role in solving the brachistochrone problem; how previous minimum problems, as for instance the classical isoperimetric problem, differ from the problem of least time descent; what Johann and Jakob Bernoulli, Leonhard Euler and Joseph Louis Lagrange meant for solutions of the new minimum problems. Since most of the beginning of the calculus of variations is based on the notion of "infinitesimal elements", in Sect. 1.3 we discuss briefly the notion of "differential" in Leibniz and Euler and, with the aim of clarifying some of the claims of the early papers of the Calculus of variations, we illustrate in Sect. 1.4 the geometrical and analytical treatment of the cycloid in the period.

Chapters 2–7 present a systematic, sufficiently complete and, we think, fair presentation of the works, actually of the mathematics in the relevant tracts of the Bernoullis, Euler and Lagrange, discussing also their connections, always being adherent to the original texts. In particular, Chap. 2 deals with the brachistochrone problem, Chap. 3 with the isoperimetric problem, according to the fundamental papers by Johann and Jakob Bernoulli. Chap. 4 deals with the beginning of the problem of finding geodesics on a surface with the contributions of Johann Bernoulli, Leonhard Euler and Alexis Clairaut. Chapters 5 and 6 deal with the key contributions of Euler to the isoperimetric problem, the former presenting the Memoirs of 1738 and 1741 that contain a famous error and the latter discussing the celebrated treatise *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentis*. Of course, we have no chance of discussing the many in specific minimum problems solved by Euler—surely one of the most beautiful and interesting aspects of the *Methodus inveniendi*—and we have to confine ourselves to discussing Euler's general method and illustrating only few examples. Finally, Chap. 7 presents the δ-calculus of Lagrange, first in the correspondence Lagrange–Euler and then in the main analytical treatise of Lagrange, adding a few more results of Lagrange that, however, belong more to the development of the calculus of variations in the nineteenth century. We conclude, in Sect. 7.5, with Euler's paper of 1771 that presents, we might say, the modern way of deriving the Euler–Lagrange equations expressing the necessary condition for minimality.

Topics in this volume were partially presented in a course–seminar held by the second author during the academic years 2011–2012 and 2012–2013 at the Scuola Normale Superiore in Pisa, dedicated to the development of calculus and mechanics in the cultural context of the eighteenth century. Expanded notes of these courses appeared as [111]. Special thanks go to friends, colleagues and students who actively participated contributing with relevant questions and very useful comments. We would like to thank particularly Vieri Benci, Sergio Bernini, Giuseppe Da Prato, Mauro Di Nasso, Marco Forti, Hykel Hosni and Massimo Mugnai. Also, we would like to thank Chiara Amadori, Federica D'Angelo, Daniela D'Innocenti and Andrea Tasini who prepared their master's theses on related topics under the supervision of the first author.

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