

Topological Fixed Point Theory for Singlevalued and Multivalued Mappings and Applications

Afif Ben Amar • Donal O'Regan

Topological Fixed Point Theory for Singlevalued and Multivalued Mappings and Applications

 Springer

Afif Ben Amar
Department of Mathematics
University of Sfax, Faculty of Sciences
Sfax, Tunisia

Donal O'Regan
School of Mathematics
National University of Ireland, Galway
Galway, Ireland

ISBN 978-3-319-31947-6 ISBN 978-3-319-31948-3 (eBook)
DOI 10.1007/978-3-319-31948-3

Library of Congress Control Number: 2016936666

© Springer International Publishing Switzerland 2016

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made.

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG Switzerland

*To my parents Fathi and Mounira
To my wife Faten and our children Hadil,
Hiba, and Youssef
and
To my brothers Imed, Aref, and my sister
Alyssa*

Afif Ben Amar

*To my wife Alice and our children Aoife,
Lorna, Daniel, and Niamh*

Donal O'Regan

Preface

Fixed point theory is a powerful and fruitful tool in modern mathematics and may be considered as a core subject in nonlinear analysis. In the last 50 years, fixed point theory has been a flourishing area of research. In this book, we introduce topological fixed point theory for several classes of single- and multivalued maps. The selected topics reflect our particular interests.

The text is divided into seven chapters. In Chap. 1, we present basic notions in locally convex topological vector spaces. Special attention is devoted to weak compactness, in particular to the theorems of Eberlein–Šmulian, Grothendieck, and Dunford–Pettis. Leray–Schauder alternatives and eigenvalue problems for decomposable single-valued nonlinear weakly compact operators in Dunford–Pettis spaces are considered in Chap. 2. In Chap. 3, we present some variants of Schauder, Krasnoselskii, Sadovskii, and Leray–Schauder-type fixed point theorems for different classes of weakly sequentially continuous (resp. sequentially continuous) operators on general Banach spaces (resp. locally convex spaces). Sadovskii, Furi–Pera, and Krasnoselskii fixed point theorems and nonlinear Leray–Schauder alternatives in the framework of weak topologies and involving multivalued mappings with weakly sequentially closed graph are considered in Chap. 4. The results are formulated in terms of axiomatic measures of weak noncompactness. In Chap. 5, we present some fixed point theorems in a nonempty closed convex of any Banach algebras or Banach algebras satisfying a sequential condition (\mathcal{P}) for the sum and the product of nonlinear weakly sequentially continuous operators. We illustrate the theory by considering functional integral and partial differential equations. The existence of fixed points and nonlinear Leray–Schauder alternatives for different classes of nonlinear (ws)-compact operators (weakly condensing, 1-set weakly contractive, strictly quasi-bounded) defined on an unbounded closed convex subset of a Banach space is discussed in Chap. 6. We also discuss the existence of nonlinear eigenvalues and eigenvectors and surjectivity of quasi-bounded operators. In Chap. 7, we present some approximate fixed point theorems for multivalued mappings defined on Banach spaces. Weak and strong topologies play a role here and both bounded and unbounded regions are considered. A method is developed indicating how to

use approximate fixed point theorems to prove the existence of approximate Nash equilibria for noncooperative games.

We hope the book will be of use to graduate students and theoretical and applied mathematicians who work in fixed point theory, integral equations, ordinary and partial differential equations, game theory, and other related areas.

Sfax, Tunisia
Galway, Ireland

Afif Ben Amar
Donal O'Regan

Contents

1	Basic Concepts	1
1.1	Topological Spaces: Some Fundamental Notions	1
1.2	Normed Spaces and Banach Spaces	3
1.3	Convex Sets	5
1.3.1	Cones	5
1.3.2	Ordered Vector Spaces	6
1.3.3	Vector Lattices	7
1.3.4	Ordered Normed Spaces	7
1.3.5	Normed Vector Lattices and Banach Lattices	8
1.4	Locally Convex Vector Spaces	9
1.5	Weak and Weak* Topologies	11
1.6	Convergence and Compactness in Weak Topologies	12
1.7	Metrizability of Weak Topologies	16
1.8	Weak Compactness in $L^1(X, \mu)$: The Dunford–Pettis Theorem	19
1.9	The Dunford–Pettis Property	23
1.9.1	Weakly Compact Operators	23
1.9.2	The Dunford–Pettis Property	24
1.10	Angelic Spaces	25
1.11	Normed Algebras	26
1.12	Measures of Weak Noncompactness	27
1.13	The Superposition Operator	31
1.14	Some Aspects of Continuity in L^1 -Spaces	32
1.15	Fixed Point Theory	35
1.15.1	The Krasnosel’skii’s Fixed Point Theorem	35
1.15.2	Leray–Schauder Theory	36
1.15.3	Multivalued Maps	37
2	Nonlinear Eigenvalue Problems in Dunford–Pettis Spaces	39
2.1	Introduction	39
2.2	Nonlinear Eigenvalue Problems	40

3	Fixed Point Theory in Locally Convex Spaces	45
3.1	Leray–Schauder Alternatives	45
3.2	Fixed Point Theory for 1-Set Weakly Contractive Operators.....	49
3.3	Fixed Point Theorems for Function Spaces	58
3.4	Fixed Point Theory for the Sum of Two Operators	61
3.4.1	Preliminaries	62
3.4.2	Fixed Point Results	63
3.5	Applications.....	72
3.5.1	A Volterra Integral Equation Under Henstock–Kurzweil–Pettis Integrability	72
3.5.2	Theory of Integral Equations in the Lebesgue Space	79
4	Fixed Points for Maps with Weakly Sequentially Closed Graph	85
4.1	Sadovskii Type Fixed Point Theorems	87
4.2	Leray–Schauder and Furi–Pera Type Theorems	88
4.3	Krasnoselskii Type Fixed Point Theorems	95
5	Fixed Point Theory in Banach Algebras	103
5.1	Fixed Point Theorems	103
5.2	Positivity	116
5.3	Leray–Schauder Alternatives	122
5.4	Applications.....	127
6	Fixed Point Theory for (ws)-Compact Operators	147
6.1	(ws) -Compact Operators	147
6.2	Asymptotic Derivatives.....	149
6.3	Quasi-Bounded Operators	150
6.4	Fixed Point Results	152
6.5	Positive Eigenvalues and Surjectivity for Nonlinear Operators	164
6.6	Applications.....	169
7	Approximate Fixed Point Theorems in Banach Spaces	173
7.1	Approximate Fixed Point Theorems	173
7.2	Approximate Nash Equilibria for Strategic Games	183
	Bibliography	187