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Objectivity, Realism, and Proof

FilMat Studies in the Philosophy of Mathematics

Springer

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To Aldo Antonelli

Preface

The philosophy of mathematics has had a tumultuous life, even when only the last one and half century is considered. While the foundational crisis between the late nineteenth century and the early twentieth century left philosophers with no clear indication of how concerns prompted by reflection on mathematics, its language, and its objects could be eventually solved, mathematics has maintained a central place in philosophical investigations. Led both by empiricist (when not sociologically leaned) approaches attempting at framing mathematics within an overall conception of natural knowledge, and by novel foundational perspectives striving to preserve an autonomous place for mathematical knowledge, the philosophy of mathematics has witnessed a growth of studies since the Seventies of the past century, and is today one of the liveliest and most stimulating areas of philosophical research, where disciplines as diverse as logic, history of mathematics, philosophy of language and science, epistemology and metaphysics find an impressively fertile common ground. While new and provoking positions have been developed on all sides of possible theoretical divides (platonist vs. nominalist, realist vs. anti-realist, empiricist vs. apriorist, philosophy-oriented vs. practice-oriented, and so on), a wealth of investigations has been flourishing, research groups gathering around specific proposals have been forming, and novel perspectives and conceptual tools have been emerging. The essays collected in this volume are meant to be a vivid example of this renewed philosophical milieu, and are the outcome of activities organized by one among the recently born international communities devoted to the philosophy of mathematics.

The background milieu for what would later become the Italian Network for the Philosophy of Mathematics, FilMat, was provided by recurrent meetings of, and collaborations between, national research groups (such as SELP and COGITO), promoted by several researchers in the philosophy of mathematics and logic based at various Universities in Italy (among which the University of Bologna, Scuola Normale Superiore in Pisa, San Raffaele University in Milan, the University of Padua, the University of Milan). Also through the continued encouragement of colleagues like Marco Panza at IHPST (Paris), and a renewed interest in publications in the philosophy of mathematics by Italian publishers (among which we would like to stress the role played by Carocci Editore) participants in those groups came to realize that their network of national and international relations should be given some more stability.

The initial suggestion for the creation of the network was then prompted by a first very successful conference that was organized at Scuola Normale Superiore, Pisa, in 2012, by Gabriele Lolli and Giorgio Venturi. The aim of that conference was to gather Italian scholars in philosophy of mathematics and closely related fields, in order to bring something to the fore: despite the philosophy of mathematics is unduly underrepresented in Italian academia, the community of Italian researchers successfully involved in the discipline is lively and conspicuous. This was not only—and not so much—meant to apply to well-known scholars already based in prestigious universities all around the world, but especially to young scholars, including PhDs and post-docs, who strive to find adequate opportunities in this field in their country, and are most of the time bound to flee towards non-Italian Universities (well beyond what is a legitimate and necessary need for international exchanges and cooperation). The aim of the Pisa conference was to make all Italian researchers, at any level, be they based in Italy or abroad, feel part of a compact and collaborative community, with clear national ties while still extremely well entrenched in international scientific research, with a significant potential for fostering successful careers of young scholars. That potential was displayed in a Springer volume that selected a number of papers from that conference, edited by Gabriele Lolli, Marco Panza and Giorgio Venturi and published in the *Boston Studies for the Philosophy and History of Science* series in 2015: *From Logic to Practice. Italian Studies in the Philosophy of Mathematics.* The volume gave a nice picture of Italian research in the philosophy of mathematics, with a special focus on the integration of logical, historical, and philosophical concerns in the philosophy of mathematics and mathematical practice.

Some of the participants in that conference—Francesca Boccuni, Gabriele Lolli, Marco Panza, Matteo Plebani, Luca San Mauro, Andrea Sereni, and Giorgio Venturi—felt the urge of providing that community with a more solid and visible platform, as a way of acknowledging the reception of Italian researchers in the philosophy of mathematics, of appreciating the successful placement of Italian scholars in international universities, of integrating younger students in a nationally based and yet geographically diffused web of professional connections, and, last but not least, of promoting the undeniable interest of studies in the philosophy of mathematics in Italian academia. Given the disseminated location of the potential members, the network form seemed most appropriate.

The FilMat Network (www.filmatnetwork.com) soon gathered a conspicuous number of affiliations of Italian scholars worldwide, counting almost 70 members at the time this Preface is written. Others may join the network in the future, and we are confident that the number of young students and early-career scholars finding in membership to the network a way of facilitating and enhancing their scientific and professional journey in this field will raise in the long run.

Despite the nation-based nature of the network, nothing is more alien to its promoting ideas and its mission than closure or isolation with respect to an international scientific community which is becoming more and more global and connected. Quite to the contrary, we believe that keeping researchers in this field in close touch despite their diffuse geographical locations is an ideal way of intensifying scientific cooperations independently of national borders, also by building on individual existing collaborations in different countries.

The FilMat Network promotes conferences, workshops, and seminars in the philosophy of mathematics and strictly related areas, also by circulating news about activities organized by its members. One of its main aims so far has been to schedule a biennial network conference. The first official FilMat international conference—*Philosophy of Mathematics: Objectivity, Cognition and Proof*—was organized by the editors of this volume at San Raffaele University, Milan, in May 29–31, 2014. As a way of stressing both the network's nation-based original inspiration and its interest in fostering international cooperation, the conference assumed a specific format. Invited speakers were selected from each of four categories: Italian scholars based in Italy (Mario Piazza, University of Chieti-Pescara), Italian scholars based abroad (G. Aldo Antonelli, UC Davies), non-Italian philosophers (Leon Horsten, University of Bristol), and early-career invited speakers (Francesca Poggiolesi, IHPST Paris). Contributed speakers were selected by double-blind review through an international call for papers.

The conference was—or so we dare say—quite a successful event. It hosted 21 talks—with the addition of an inaugural lecture by Stewart Shapiro (Ohio State University)—and received 38 submissions from 36 international universities and research institutions from Austria, Belgium, Brazil, Bulgaria, Finland, France, Germany, Hungary, Israel, Italy, Poland, Portugal, UK, and USA. Considering joint works, submissions came from a grand total of 42 authors, of which 9 Italy-based and 6 non-Italy-based Italian nationals, and 27 international authors. Among all of them, 18 were young scholars and early-career researchers. Numbers proved the format to be successful, and it is likely to be preserved in the future. The second FilMat conference has taken place at the University of Chieti-Pescara in May 26–28, 2016, organized by Mario Piazza together with Pierluigi Graziani (University of Urbino) and Gabriele Pulcini (University of Campinas), and statistics of submissions in terms of quantity and international provenance equalled those of the first conference.

We are grateful to Springer, to the Editors of the *Boston Studies in the Philosophy and History of Science* series, and in particular to Springer's publishing editor Lucy Fleet for having accepted and supported our proposal of making a volume out of selected contributions from the FilMat 2014 conference, and to the Project Coordinator Karin De Bie, as well as to Steve O'Reilly and Gowtham Chakravarthy for their support in the production process. We also wish to thank two anonymous reviewers for their precious comments on the book proposal and final draft. Together with the collection stemmed from the Pisa conference, we believe this volume will testify the value of the kind of research in the philosophy of mathematics that has been gathering around activities of the FilMat Network, and we hope other volumes will follow suit.

Above all, we are grateful to all the authors who submitted to this volume for their cooperation and patience in going through a double-blind review process, which has often led to significant modifications and improvements of contributions based on the valuable suggestions from a panel of about 30 international reviewers, to whom we express our warmest thanks. Given changes occurred along this process, and the final distribution of themes across contributions, the title of this book has been slightly changed from the title of the original conference where most papers were originally presented.

We also would like to thankfully acknowledge the support of the PRIN Italian National Project *Realism and Objectivity* (national coordinator: Pasquale Frascolla, Basilicata University), and in particular to the San Raffaele research unit *Cognitive Sciences and Scientific Objectivity* (unit coordinator: Claudia Bianchi, San Raffaele University).[1]

Of all authors who directly or indirectly supported our project—by participating in the conference or submitting their contribution to the volume—this collection is dedicated to G. Aldo Antonelli. Aldo prematurely and unexpectedly passed away on October 11, 2015. He was the nicest person and an outstanding philosopher of mathematics and logic. His death was an immense loss to the scientific community as a whole. Aldo was extremely supportive of the network project since we first invited him as a network member and as a speaker to the FilMat conference in Milan, and maintained his unshaken support for this volume. When he passed away, he was just about to send us a revised version of his paper. Reviewers suggested just minor revisions, and we then decided to publish the paper as it was. In their comments reviewers clearly manifested sincere appreciation for Aldo's paper. Sean Walsh emphasized that "the paper was eloquently composed and a joy to read." Roy Cook stressed that "the paper is excellent," and accompanied his report with the following confidential comment:

> Of course, about halfway through the paper I also became pretty confident of the identity of the author, and if I am right, then my positive report is not surprising: the person who I suspect wrote the paper usually produces excellent work that rarely needs significant modification or revision! (Of course, I could be wrong about who the author is, in which

[1]Even though this volume is a self-standing enterprise, the FilMat conference that made it possible received financial support from various sources. Besides the PRIN project, we take the opportunity to thank again the Ph.D. Program in *Cognitive Neurosciences and Philosophy of Mind* (San Raffaele University/NeTS at IUSS Pavia); the Ph.D. Program in *Philosophy and Sciences of the Mind* (San Raffaele University); SELP (Seminario di Logica Permanente). The conference was held under the auspices of AILA (Italian Association for Logic and its Applications), SIFA (Italian Society for Analytic Philosophy), SILFS (Italian Society for Logic and Philosophy of Science), and in collaboration with COGITO Research Centre (Bologna) and CRESA Research Centre (San Raffaele). Special thanks go to the then Dean of the Faculty of Philosophy at San Raffaele University, prof. Michele Di Francesco, for his support in promoting this and many other activities in the philosophy of mathematics.

case the moral is that the actual author of the paper commendably met the very high standards that are typical for the person who I have in mind).

We are thankful to them for their permission to disclose their names and report parts of their comments as a way of witnessing once more the excellent quality of Aldo's research. We are all the more grateful to Aldo's partner Elaine Landry and his son Federico Antonelli for having made the publication of his paper possible. Aldo was the clearest representative of the kind of scholars the network is thought for, and we are proud to have been given the chance of meeting him and collaborating with him. He was organizing a workshop on *Ontological Commitment in Mathematics* together with Marco Panza, to be held at IHPST in Paris, where he would have presented the paper published in this volume. The workshop was turned into an event *in memoriam of Aldo Antonelli* and took place in Paris on December 14–15, 2015. Andrew Arana, in collaboration with Curtis Franks, delivered a memoir of Aldo's life and work, which they kindly gave us permission to include in this collection. As a way of homaging Aldo's work, together with his partner and his colleagues and friends Robert May and Marco Panza, we decided to include also a discussion note summarizing and systematizing the discussion that took place after Aldo's paper was read at the Paris workshop, which Robert and Marco kindly agreed to edit. We are confident that this discussion will do nothing more than stressing once more how stimulating and thought-provoking Aldo's work in the philosophy of mathematics can be.

If something has to be witnessed by the papers included in this collection, it is the variety of philosophical concerns that may be prompted by current reflection on mathematics. It goes without saying that what is offered here is a necessarily partial picture—as the vast production of papers and books in this field in recent years, supported by the creation of dedicated networks and research groups, testifies. As Stewart Shapiro emphasizes through the consideration of three case studies, there is a variety of stimulating ways in which mathematics and philosophy can reciprocally contribute to an improved understanding of their respective fields. Of these interactions, and more generally of the philosophical concerns that mathematics raises, three are the main areas on which the papers collected here focus, briefly codified in the three key notions in the title: *Objectivity, Realism, and Proof.*

How a pivotal area of our rational life can be granted the objectivity it deserves is a classical problem in the philosophy of mathematics, which becomes extremely pressing when the shadowy nature of its objects is considered and their connection with the concrete, empirically accessible world is investigated. Essays in Part I (*The Ways of Mathematical Objectivity: Semantics and Knowledge*) are all, to different extents, bearing on these issues. Fregean and neo-Fregean philosophy of mathematics attempted to assuage similar concerns by appropriate semantic analysis of mathematical discourse, but shared solutions are a long way off. Aldo Antonelli (as also shown in the Discussion Note of his contribution edited by Robert May & Marco Panza) and Robert Knowles both confront semantic issues concerning mathematical discourse in a Fregean framework. On a different but related note, mathematics can at the same time be thought of as being constituted by *a priori* truths and as both biologically grounded and entrenched in practice and applications. Markus Pantsar and Marina Imocrante investigate in various ways how the alleged *a priori* character of pure mathematics can be integrated with either an empiricist framework informed by cognitive sciences or an epistemology of mathematics especially focused on applications and actual practice.

Objectivity seems assured when mathematics is considered as a discourse about a well-defined realm of objects, which mathematical theories are supposed to describe. However, the nature of different mathematical objects and the structure of the mathematical universe come in a variety of shapes. Essays in Part II (*Realism in a World of Sets: From Classes to the Hyperuniverse*) focus on these issues, with a particular attention to that essential domain of mathematical objects which sets are. Leon Horsten, Brice Halimi, and Gianluigi Oliveri discuss different approaches to the nature of sets, by investigating respectively the import of conceptions of the infinite on a characterization of classes, the relationship between sets and categories, and the conception of set theory as a science of structures rather than individual objects. The remaining essays in this Part, on the other hand, are concerned with finding an adequate picture of the set-theoretic universe, by connecting a realist picture with a pluralist conception of the set-theoretic domains. Claudio Ternullo & Sy-David Friedman, Neil Barton, and Giorgio Venturi all explore, through different approaches, a conception of the set-theoretical universe today known as multiverse, respectively by relating it to the so-called Hyperuniverse program, by investigating the extent to which relativism may be acceptable in a conception of the set-theoretical domain, and by considering how techniques like forcing may support one or another realist view of such domain.

Both the problems of objectivity and realism need to face an undeniable fact: even when it is understood as aiming at a faithful description of an independent realm of mathematical objects, mathematics is a human activity, where the goal of attaining truth is pursued through symbolic languages by regimentation and clarification of more or less informal notions in appropriate formal systems. Essays in Part III (*The Logic Behind Mathematics: Proof, Truth, and Formal Analysis*) offer new perspectives on some classical issues in this vicinity. Contributions by Mario Piazza & Gabriele Pulcini, and Carlo Nicolai, both deal with specific issues concerning truth in formal theories, either as related to our access to the truth of Gödel's sentence $\mathcal{G}$, or as related to the relationship between axiomatic truth theories and comprehension axioms. In the last three essays, the interplay between some central notions in mathematics and metaphysics (including the metaphysics of mathematics) and their proper formalization is explored by Francesca Poggiolesi, Massimiliano Carrara & Enrico Martino & Matteo Plebani, and Samantha Pollock, either by investigating how the proper logic underlying the epistemic and metaphysical notion of grounding should be made precise, or by suggesting that a primitive notion of finiteness may be essential to singling out the standard model of arithmetic, or finally by exploring how informal beliefs may be involved in the appreciation of technical results such as categoricity theorems.

Through their diverse approaches and focus, the essays in this volume collectively prove once more how rich and stimulating mathematics can be for philosophy on its semantic, epistemic, and ontological aspects. They offer novel perspectives on vexed theoretical issues and promise to deepen our understanding of such a fascinating part of human thought like mathematics is. We are confident that they will stimulate further discussion and will greatly contribute to current debates.

Francesca Boccuni
Andrea Sereni

Contents

Objectivity, Realism and Proof in the Philosophy of Mathematics: An Introduction

Francesca Boccuni and Andrea Sereni

Philosophy, Mathematics, and the Philosophy of Mathematics

Since ancient times, mathematics has always been a source of fascination and philosophical reflection. It has traditionally been considered the major example of an area where knowledge can achieve the certainty and exactness many have seen as a human epistemic ideal. Its pervasiveness and usefulness in everyday and scientific applications have made it the prime tool for the study of physical reality. At the same time, both the allegedly heavenly nature of its objects and its dealing with the infinite have become a constant challenge for mundane and finite beings as we are. During centuries, and especially since the nineteenth century thanks to the development of modern logic, mathematics has stopped being just a source of philosophical concern, and has become a powerful instrument in testing philosophical theories on meaning, knowledge, justification (in the form of logical and mathematical proof), and ontology. The philosophy of mathematics has long become a well-defined and still multifaceted area of philosophical investigation, thanks to its many connections with disciplines such as the philosophy of language, logic, epistemology, and metaphysics, not to mention logic and the history and practice of mathematics itself.

Especially during the past century and a half (although in a way that has its roots in views whose development has taken millennia), positions advanced in the philosophy of mathematics have tended to crystallize in a number of oppositions. We

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find platonists believing in the existence of abstract mathematical entities, opposed by nominalists denying such existence and trying to bring mathematics back to a worldly affair. We find realists believing in the objective mind-independence of the truth-values of mathematical statements, opposed by various sorts of anti-realists or constructivist views, together experimenting variations on the somewhat *cliché* metaphorical dispute between the mathematician as an inventor and the mathematician as an explorer. We find rationalists of sorts trying to secure the *a priori* status of mathematical knowledge, opposed by various brands of empiricists trying to ground mathematics on empirical evidence. We have (or have had, at least) foundationalists looking for the basic bricks of the mathematical edifice, and anti-foundationalists approaching mathematics as a fallible, when not sociologically determined, practice.

In recent times, while some of these oppositions are still standing, much more nuanced approaches have in fact developed. Suffice to think of the sharper tools that empirical findings have offered to long neglected empiricist insights, or even more of the exploration of fruitful connections between traditional philosophical concerns and a more attentive study of mathematical practice (on which also a volume somehow precursor to this FilMat collection is chiefly devoted; cf. Lolli et al. 2015). And yet, many of those antitheses still underlie much of current research. Others can be thought of, also crossing departmental boundaries. One could think, for example, of the contrast between a primacy of philosophical analysis of intuitive notions on the one side, and, on the other, the role of regimentation in formal languages as a means sometimes to explain, other times to replace those intuitive, informal notions. Again, one can think of the opposition between monistic attitudes towards the logic underlying our mathematical reasoning as opposed to more liberal pluralistic approaches. Prompted by significant technical results, this very same rivalry can arise even in different conceptions of the mathematical universe, or universes, itself.

Focusing on three key concepts in the philosophy of mathematics, *objectivity*, *realism,* and *proof*, the essays collected in this volume offer new perspectives on how theories on each sides of those divides can either be defended or opposed, or else on how middle grounds between them can be found. How is the objectivity of mathematics to be secured, even when its subject matter is not given a platonist construal? What semantic analysis should model mathematical meaning when we renounce thinking of mathematical terms as picking out mathematical objects in some univocal and determinate way? Which role should the cognitive roots and the empirical applications of mathematics play in achieving such objectivity? What are the objects, if any, inhabiting the universe of such a fundamental domain of mathematical discourse as that modeled by set theory? And to what extent should we conceive of it as a unique universe, rather than a plurality of universes each regulated by its own axioms? What is the relation between our intuitive conception of truth in mathematics and the limitations imposed by the formal systems possibly required to give rigor and precision to that conception? And more generally, how does formal treatment affect our understanding of many informal notions which seem to regulate mathematical thought?

These are but some of the questions that are raised, and to which insightful answers are offered, in the essays composing this volume. It goes without saying that all the themes that are explored here are nothing but a sample of the amazing richness of the philosophical concerns that are prompted by reflection on mathematics, not to mention the powerful tools that mathematics (and mathematical logic) can offer to philosophical analysis in so many areas.

A vivid picture of this close interaction between mathematics and philosophy is offered by **Stewart Shapiro**'s stimulating opening contribution, *Mathematics in Philosophy, Philosophy in Mathematics: Three Case Studies*. Shapiro first reminds us of the origins of the philosophical fascination of philosophy with mathematics, tracing it back to its roots in ancient Greek thought. By rehearsing Plato's progressive distancing from Socratic method towards the exactitude of thought aided by and modeled on mathematics, we are reminded of how mathematics (including geometry) becomes in Plato's views an essential prerequisite to gain any intellectual advancement, a means to achieve objective knowledge and to draw the soul "from the world of change to reality." Plato's views on mathematics would become essential to theoretical oppositions (rationalists vs. anti-rationalists, platonists vs. anti-platonists, etc.) which are still with us today.

Greek thought is also the culprit of Shapiro's second case study, the birth and growth of logic as a means for understanding thought through what can be seen as a precursor of a regimented formal language in Aristotle's syllogistic. Quickly following the development of logic up to modern times, we are then presented with a nice contrast. On the one hand, the interplay between the exactness of logico-mathematical language and the analysis of ordinary language becomes pivotal in the development of mathematical logic since the end of the nineteenth century, suggesting both that ordinary language could be successfully regimented and controlled, and that mathematics can be the sharpest litmus paper for testing philosophical theories about meaning and knowledge. On the other hand, the possible mismatch between the exactness of logical tools and the possibly loose discourses they are used to systematize became apparent in the analysis of vague terms. Shapiro reminds us of how careful we need be when we use sharp tools to analyze fuzzy subjects, and discusses the effect this warning has had on others' and his own theory of vagueness.

Like vagueness, other paradox-threatening areas of discourse are subject to different attempts at rigorous regimentation by logico-mathematical tools. One is continuity, which underlies paradoxes of motion since antiquity and is central to the development of modern mathematics. As his final case study, Shapiro discusses several conceptions of the continuum, from Greek to modern times, highlighting how they may suggest revisions of our background logic from classical to non-classical ones, and briefly sketches the view Geoffrey Hellman and himself are developing. According to this view, there is no monolithic notion of continuity that can completely capture the intuitive conception we may have of it: there are rather several precisifications, which can be brought to light once the informal notion is regimented through different formal tools.

Shapiro's conclusion is a neat example of how logico-mathematical language can help in the formal analysis of philosophical (and mathematical) notions, and directly connects with the papers in Part III of this volume. More generally, it is easy to see how this opening essay paves the way for the main themes of the contributions to follow: the problem of accounting for mathematical objectivity over and above a commitment to mathematical objects, the problem of giving a satisfying characterization of the mathematical universe, and the problem of adequately capturing informal notions through logical and formal tools. In the rest of this introduction, we will see how all these themes are explored by the authors contributing to this volume, how they deepen our understanding of the relation between philosophy, mathematics, and logic, and suggest novel directions of inquiry for contemporary philosophy of mathematics.

The Ways of Mathematical Objectivity: Semantics and Knowledge

The essays in Part I of this volume all deal, from rather different perspectives, with issues concerned with the objectivity of mathematics (while issues directly pertaining with mathematical realism are postponed to Part II). Despite their difference in focus, they manifest a minimal shared target: they all suggest ways in which it possible to account for the objectivity of mathematics even without endorsing a platonist conception of its subject matter. To this aim, the first two essays focus on the semantics of mathematical discourse, whereas the second two essays explore problems in accounting for knowledge of both pure and applied mathematics. Together, they offer a nice picture of how reflections on meaning and epistemology in contemporary philosophy can offer new perspectives on account of mathematical objectivity.

The Semantics of Mathematical Discourse

It is has become a somewhat abused habit to begin discussions of the relationship between realism and objectivity by quoting the old dictum attributed to Georg Kreisel according to which the central concern in the philosophy of mathematics is not the existence of mathematical objects, but rather the objectivity of mathematical truths. If we indulge in this habit it is not (just) for lack of imagination, but mostly because the first essay in this section is a compelling contemporary example of how that dictum can be given new life.

It is a well-known component of Frege's philosophy of mathematics that numbers are to be conceived as abstract objects, more specifically a particular kind of logical objects, i.e., extensions of concepts. Since (at least) Benacerraf (1973), platonism has been on the receiving end of a powerful criticism: it is hard to reconcile its answer to the question "what are mathematical objects?" with a plausible answer to the question "how can we have knowledge of them?"

Neo-logicists like Crispin Wright, Bob Hale and collaborators—who preserve the platonist component of Frege's philosophy, freed from any appeal to extensions (cf. Hale and Wright 2001)—have resorted to a suitable epistemology of abstraction principles in order to account for how knowledge of natural (finite cardinal) numbers is achieved. Still, some view the appeal to abstract entities, even when made in the context of consistent principles such as Hume's Principle, as unwelcome. **Aldo Antonelli** also takes it as unnecessary. Building on previous works (see e.g., Antonelli 2010a, b, 2013), in his paper *Semantic Nominalism: How I Learned to Stop Worrying and Love Universals* he offers a naturalistic view of abstraction principles which is claimed to be epistemically more adequate than the neo-logicists' one. In his view, abstraction principles do not provide us with a particular system of objects (cardinal numbers), but rather with representatives for equivalence classes of second-order entities. Such representatives will be available provided the first- and second-order domains are in the equilibrium dictated by the abstraction principles, but otherwise the choice of representatives is unconstrained. Under this conception of abstraction, abstract entities are the referents of abstraction terms: such a referent is to an extent indeterminate, but we can still work with such terms, quantify over their referents, predicate identity or non-identity, etc. Our knowledge of them is limited, but still substantial. In particular, we know whatever has to be true no matter how the representatives are chosen, i.e., what is true in all models of the corresponding abstraction principles. We won't know anything about the special nature of the representatives. But we will know whatever follows from the positing of such representatives. Two remarkable features of this proposal should be noticed. First, it is backed up by an "austere" conception of universals, according to which these are first-order objects, i.e., ways of collecting first-order objects. In this, Antonelli builds on Dedekindian suggestions in order to claim that second-order logic does not by itself import any novel ontological commitment over and above an ontology of first-order, naturalistically acceptable, objects. Second, it is a striking outcome of Antonelli's proposal that, in the case of arithmetic, even if Hume's Principle is understood under the construal described above, the proof of Frege's Theorem (the derivation of second-order Peano Axioms from full impredicative second-order logic with the sole addition of Hume's Principle) still goes through unaffected, since there is nothing in the proof that depends on an account of the "true nature" of numbers. Thus, we are left with a viable construal of logicism, given by the combination of semantic nominalism and a naturalistic conception of abstraction. It goes without saying that this proposal is thought-provoking and open to developments and objections, as the lively discussion of Antonelli's paper held by a number of scholars at a recent Paris workshop on *Ontological Commitment in Mathematics* (IHPST, December 14–15, 2015) and reported in the *Discussion Note* edited by **Marco Panza** and **Robert C. May**, clearly displays.

Antonelli's view is moved, at least partly, by concerns with an ontology of abstract mathematical objects which seems to be suggested by a certain reading of abstraction principles. The same platonist ontology seems suggested by a rather natural semantic analysis of ordinary attributions of numbers, i.e., those statements like “The number of planets in the solar system is eight” (*Zhalangaben*, in Frege’s original language). Following Frege, this analysis interprets such statements as identity statements of the form “The number of planets in the solar system = 8.” The same holds for statements attributing measures like “The mass of Jupiter in kilograms is 1.896 x 10^{27}.” In his paper *Semantic Assumptions in the Philosophy of Mathematics*, **Robert Knowles** discusses and rejects this analysis. Rival accounts of this kind of sentences have been offered (e.g., by Hofweber 2005 and Moltmann 2013). Knowles considers a wide range of linguistic evidence and finds these alternative approaches defective. Any suitable analysis must account for the fact that numerals (and expressions for magnitudes) can occur in natural language both in substantival and adjectival position. Only the former can lend support to a realist conception of mathematical objects, since the semantic function of adjectival expressions is not to pick out singular objects. Frege famously held that sentences in which such expressions occur in adjectival position can always be transformed in sentences in which they occur in substantival position, but many authors have found this claim questionable. Knowles agrees with views rival to Frege’s that the pre-copular expressions in sentences such as the above are not referring expressions. Building on an account of interrogatives, he suggests that a sentence like ‘The number of planets in the solar system is eight’ is true if and only if the fact that uniquely and exhaustively answers the question ‘How many planets are there in the solar system?’/‘What is the number of planets in the solar system?’ is identical to the fact that there are eight planets in the solar system. Likewise for attributions of measures. Thus specified, the truth conditions of these sentences do not seem to involve mathematical objects, but only certain kinds of facts. By itself, then, their truth cannot underlie any realist argument concerning mathematics objects, at least not until either evidence from the semantics of other mathematical sentences is offered, or additional arguments are presented to the effect that it is in the very nature of the facts making those sentences true that mathematical objects are among their constituents.

Mathematical Knowledge, Pure and Applied

Knowles’ discussion suggests that, at least in so far as we focus on statements of applied mathematics, there may be (linguistically adequate) ways of accounting for their objective truth-value which do not resort to any appeal to the role of abstract mathematical objects. The problem of salvaging the objectivity of mathematics while renouncing the natural and yet problematic picture of an alleged acquaintance with mind-independent shadowy abstract entities has to account for at least two salient aspects of mathematics. First, mathematics is a human activity deeply entrenched in our basic cognitive capabilities; and second, one major reason to believe in the truth of mathematics comes from its impressive record of successful applications. The next two essays in this section focus on these two aspects of mathematics respectively.

In *The Modal Status of Contextually A Priori Arithmetical Truths*, **Markus Pantsar** discusses how some crucial epistemic features of arithmetical knowledge, such as aprioricity, objectivity, and necessity can be accounted for in a view of mathematics that gives justice to its biological roots. An "empirically feasible" philosophy of arithmetic (cf. Pantsar 2014), according to Pantsar, sees it as stemming from a bundle of biological primitives regulating basic numerical skills (as recently detailed by findings in the cognitive sciences), a "proto-arithmetic" which later develops into actual mature arithmetic thanks to the role of language and the understanding of a successor operation. This empirical conception of arithmetic can dispense with an appeal to abstract objects, and still claim for an *a priori* character of arithmetical knowledge. Arithmetical knowledge is explained to be "contextually *a priori*": once empirical facts determine a particular context, arithmetic is *a priori* in so far as its methodology is detached from those of empirical sciences and its subject matter is not given by the psychological processes of mathematical reasoning. Biological primitives underlying mathematical knowledge also constrain our ways of experiencing the world in a way that seems to afford the required objectivity (or "maximal intersubjectivity") of arithmetic. Pantsar then focuses on arguing that in his framework arithmetical truths come out as necessary too. Since such truths are grounded in biological primitives of contingent beings, it seems they cannot be true in all possible words, and thus numerical terms cannot function as rigid designators. However, they can rigidly designate those concepts which are developed through our biological settings in all those possible worlds which are inhabited by sufficiently developed biological beings, thus picking out the same thing in all those worlds in which that thing exists. And this, argues Pantsar, is enough to bestow on arithmetical truths their necessary modal status.

While Pantsar deals with reconciling traditional epistemic features of arithmetic with a cognitively informed account of its development, **Marina Imocrante** focuses on how accounts of applied mathematics can foster our understanding of mathematical knowledge. In her *Epistemology, Ontology and Application in Pincock's Account* she critically discusses one of the most developed proposals for an epistemology of applied mathematics, the structural account advanced by Pincock (2012), according to which, roughly, applications of mathematics can be explained through structural relations (mappings or morphisms) between physical systems and suitable mathematical structures. In Pincock's account, applications can be explained with no prior commitment to the ontology of pure mathematics, which has to be decided independently. However, according to Imocrante, Pincock makes a number of assumptions that together threaten to make his account unstable. On the one hand, he offers an "extension-based epistemology" for mathematical concepts, which is meant to take into proper account the historical development of such concepts in mathematical practice. On the other hand, he recommends that pure mathematical statements should be justified *a priori*. Pincock couples these claims with the adoption of a form of semantic realism for mathematical statements and a form of semantic internalism for mathematical concepts. Imocrante suggests that the latter seems to stand in contrast with the aforementioned extension-based epistemology, and should rather be replaced by a form of externalism about mathematical concepts. As a consequence, the structural account would be dispensed by the need of requiring a *a priori* justification for mathematical statements. In Imocrante's account, however, semantic externalism for mathematical concepts does not entail commitment to a form of ontological realism about mathematical objects. On the contrary, Imocrante suggests possible ways in which such externalism can be made consistent with a "world-driven" understanding of mathematical concepts as determined by contingent facts in the history of mathematical development, consistently with the extension-based epistemology underlying the structural account of applied mathematics.

Again, mathematical discourse seems to gain the required objectivity in ways that do not require thinking of mathematics as a true description of a realm of *sui generis* abstract mathematical objects. All essays in Part I offer different perspectives on how to reach this aim, through both semantic and epistemological analysis of pure and applied mathematics. However successful, all these views have to cope with the problematic but still very natural intuition that it is such a domain of immaterial objects that mathematical theories may be about. While many believe that there are various routes to avoid similar commitments in the case of the theories of natural and other numbers, it is much harder to demise the realist picture when we come to more fundamental theories such as set theory. The essays in Part II are devoted to explore different aspects and diverging conceptions of how the universe of sets may nonetheless be characterized.

Realism in a World of Sets: From Classes to the Hyperuniverse

Even when the picture of mathematics as a discourse aimed at a true description of a self-subsistent, mind-independent domain of abstract objects is endorsed, it is far from clear how such a domain should be thought of. Several reductive strategies may be available (be they successful or not) when it comes to objects like natural or real (or complex, for what matters) numbers. In an Ockhamist spirit, some may want to impoverish the apparent abundance of the mathematical universe to a single kind of fundamental objects, to which others can be suitably reduced. Set theory has historically been playing the role of such mathematically and ontologically fundamental theory. Still, a proper understanding of both these reductive strategies, and of such fundamental domain of mathematical objects, requires us to have a clear picture of how the set-theoretic universe is structured. Several problems opens up here, different sets of set-theoretical axioms can be explored and proposed through different strategies, relations between sets and other mathematical objects like classes, categories and structures may be assessed, diverging conceptions of the set-theoretic universe may be forthcoming, and it may even turn out that realism about such a unique universe delivers a too simplistic picture, which should give way to some sort of pluralist conception. The first three essays in Part II focus on crucial problems concerning the notions of class and absolute infinity, the relation between sets and categories, and a platonist versus a structuralist conception of the subject matter of set theory. The remaining three essays deal with pictures of the set-theoretic universe alternative to the monist realist one, in conceptions known as multiverse and hyperuniverse, and with how new axioms should be justified in developing such a plurality of universes.

Varieties of Mathematical Objects: Classes, Categories, and Structures

Are there collections beyond sets? As **Leon Horsten** reminds us in his *Absolute Infinity in Class Theory and in Theology*, Zermelo thought that there are none, that the set-theoretic universe is made of a potentially infinite hierarchy of so-called normal domains, and that the set-theoretic universe itself cannot be a completed collection, cannot be considered as a set and cannot be quantified over. Opposite to Zermelo's view seems to stand Cantor's conception of the set-theoretic universe as a completed absolute infinity. Horsten notices how Cantor's conception of absolute infinity recalls some conceptions of the infinite in Western theology. The role of reflections principles underlies this analogy: once the whole set-theoretic universe can be reflected and represented by some collection at a lower stage, it becomes somehow ineffable, and ineffability is one of the traits of absolute infinity in theology. Indeed, Cantor himself is not clear on whether his conception of the absolutely infinite is theological or mathematical in nature. Horsten explores the tensions in Cantor's conception, and possible different interpretations of his view, but aims at rehabilitating it in contrast with Zermelo's. If we adopt Cantor's view that the universe is a completed whole and acknowledge classes beyond sets, Horsten shows, we can motivate stronger reflections principles than what is allowed in Zermelo's framework, like the Global Reflection Principle (*GRP*). This very roughly states that the whole universe with its parts is indistinguishable from some initial set-sized cut of itself and its parts. And this idea closely resembles Philo of Alexandria's view that "there are angels such that every humanly describable property of God also applies to them." After advocating a mereological conception of proper classes, Horsten distinguishes between a mathematical global reflection principle, $GRP_{\Sigma^0_\infty}$, where only quantification over sets is allowed, and a mereological one, $GRP_{\Sigma^1_\infty}$, where also quantification over proper classes is allowed. In the resulting picture, "good non-theological sense" is made of Cantor's picture of the set-theoretic universe. This view not only allows for stronger reflection principles and large cardinal axioms, but also allows claiming that such principles are "intrinsically motivated" by the same pattern of reasoning that justifies analogous principles concerning the absolutely infinite in theology.

Horsten's essay presents us with competing conceptions of how the cumulative hierarchy of sets could be conceived and the axioms describing it justified. As we will see in the rest of this part of the volume, several pictures of the set-theoretic universe can be contrasted. But one may also wonder whether sets are indeed the right entities to play the crucial foundational role they have been long thought to play. Traditionally, that role has been challenged by category theory (cf. MacLane 1978). As **Brice Halimi** initially reviews in his *Sets and Descent*, a host of arguments have been offered to decide the rivalry between sets and categories (from claiming that categories, as collections of objects plus collections of arrows, presuppose set theory, to claiming that *ZFC* is nothing but by a particular case of set theory that can be developed using an elementary theory of the category of sets). However, Halimi's intent is not to have a final say on the dispute, but rather to assume a mathematically informed attitude through close looks at mathematical practice, and to investigate when set theory and category theory can combine in fruitful ways, well beyond the adjudication of the foundational primacy to either. Halimi focuses on one relevant instance of this possible interaction (whose technical details are perspicuously explored in the course of the exposition): Algebraic Set Theory (AST), a reconstruction of *ZFC* in category-theoretic terms inspired by algebraic geometry (cf. Joyal and Moerdijk 1995). More specifically, Halimi shows how AST is guided by descent theory, a theory coming from algebraic geometry which studies the shift from local data to a global item through a "glueing" procedure. Halimi first introduces the framework of fibered categories (a category-theoretical generalization of the notion of surjective maps), a notion on which descent theories relies. He then proceeds by showing how the first axioms of AST combine the respective frameworks of both *ZFC* and descent theory, and concludes by stressing how AST neatly displays a fruitful interaction between the two theories. In a nutshell, AST uses a fibered category in order to reinterpret *ZFC* as an arrow-based theory and to enrich it with the geometric ideas of localization and glueing. So the way in which AST exploits both category and set theory is grounded in techniques coming from abstract algebraic geometry and algebraic topology. Beyond exemplifying a nice cooperation between the two rival theories in the foundation of mathematics, then, AST also has the advantage of linking those foundational theories to other crucial portions of mathematical practice.

Whether or not one follows Halimi's advice of focusing more on the interaction than on the foundational rivalry between sets and categories, it is undeniable that finding a proper characterization of the set-theoretic universe has had a pivotal function in essentially foundational projects. The pull towards the intuitive conception of that universe as a unique totality (which may not itself form a definite collection) is admittedly strong, and reinforced by views of that universe (as the one due to Cantor) that we have already seen explored by Horsten. According to what **Gianluigi Olivieri**'s discussion in *True V or not True V, That is the Question*, this static picture of the universe is wrong: there is no such thing as the one true universe of sets. According to Oliveri, this naïve idea collapses as soon as one investigates further the accompanying thought that set theory is a science of objects. This generates a two-horn dilemma. On the one hand, if first-order *ZFC* is consistent, and *V* is a model of it, then we are left with the consequence (by Gödel's theorems, Löwenheim–Skolem theorem, and forcing techniques) that there is a plurality of even non-isomorphic such models, and we are at a loss in individuating the one true universe. On the other hand, if we renounce the idea of a one true universe *V*, we are, according to Oliveri, bound to adopt views (such as constructivism or Meinongianism about set-theoretic objects) that negatively affect both our treatment of independent questions like *CH* and the foundational role of set theory more generally. Oliveri's way out of the dilemma consists in rejecting the idea that mathematics is a science of objects in favor of the view that mathematics is a science of structures. This latter view finds unacceptable the very idea of a universe as the totality of all sets and, therefore, is against the idea of a true *V*. It also upholds metaphysical realism about structures, though it does not do away with mathematical objects, but merely restricts mathematical investigation to the study of structures. Contrary to what happens with other views renouncing the idea of a one true universe, however, it is committed to realism about truth-values of mathematical statements. This picture of the subject matter of set theory, in the end, is intended to oppose the "architectonic metaphor" which, according to Oliveri, underlies so much of the discussion of set theory as a foundation of the edifice of mathematics, offering a much more nuanced view of foundational issues in mathematics, coherently with a view of the latter as a partly fallible and conjectural discipline, which Oliveri motivates and supports.

Varieties of Mathematical Universes: Multiverse and the Hyperuniverse
The first three papers in Part II raise specific questions concerning the universe of sets conceived as the single domain of a foundational theory: whether it should contain only sets or also classes when accounting for absolute infinity, whether it should be conceived algebraically leading to new perspectives on the relations between sets and categories, and whether such domain is properly characterized as being a domain of objects and not of structures instead. The last among these essays suggested that the question of what is the one true set-theoretic universe may be an ill-posed question. Without abandoning a conception of set theory as a science of objects, the authors of the remaining three papers in Part II may share the same concern. As a consequence, they explore different views of the nature of the set-theoretic universe, or universes. Multiversism and the Hyperuniverse picture building on it suggest that different systems of objects obtained by variously interpreting the axioms of set theory could be taken to constitute a plurality of distinct and still coexisting universes. When further investigated, this conception elicits a vast array of concerns for any realist attitude towards sets, and for our understanding of axioms as basic descriptions of a univocal domain of objects.

In their contribution *The Search for New Axioms in the Hyperuniverse Programme*, **Sy-David Friedman** and **Claudio Ternullo** explore a novel procedure for the search of new intrinsically justified axioms in the Hyperuniverse program recently developed by Arrigoni and Friedman (2013). The authors first distinguish between potentialist and actualist conceptions of the set-theoretic universe, and review Zermelo's conception as being potentialist in height (the height of *V* is not fixed and new ordinals can always been added) and actualist in width (the width of *V* is fixed and no new subsets can be added at each stage). This conception gives rise to a form of vertical multiverse that only partially (in its height dimension) satisfies a principle of plenitude which the authors take as underlying the iterative conception of set, according to which "given a universe of sets, all possible extensions of it which can be formed are actually formed." The Hyperuniverse ($\mathbb{H}^{ZFC}$) is a conception of the set-theoretic universe which is meant to allow for maximal extendibility in both dimensions: it is the collection of all countable transitive models of *ZFC*. Friedman and Ternullo motivate the restriction to such models and explore the underlying logic in which satisfaction in this multiverse can be defined. They then move on to assess new candidate axioms, which now take the form of higher-order maximality principles about V, formulated in a Zermelian framework, satisfied by members of $\mathbb{H}$, and coming with a stock of associated first-order consequences. Rival conceptions of what new set-theoretical axioms should be are then discussed. As far as the background ontology is concerned, Friedman and Ternullo call their view "dualistic," meaning that they endorse elements of both monism and pluralism about the set-theoretic universe: they postulate one single, maximally extendible universe, but they also countenance different universes given by the relevant models, where new set-theoretical truths can be detected.

Friedman and Ternullo are chiefly moved by epistemological concerns. They want to suggest novel evidential paths to secure truth of new axioms without endorsing any pre-formed ontological picture. Accordingly, their view is more focused on how the concept of set should be adequately cashed out, rather than with how a particular view of set-theoretic ontology should be motivated. It goes without saying, however, that the multiverse picture raises substantial concerns both at ontological level and at the level of the semantics of set-theoretical discourse. In his *Multiversism and Concepts of Set: How Much Relativism is Acceptable?*, **Neil Barton** focuses on the semantic problems. While agreeing that the multiverse picture as advanced by Hamkins (2012) seems to square quite nicely with mathematical practice in set theory, Barton finds it problematic at a more philosophical level. He suggests that this kind of multiversism can either be interpreted as providing an ontological view, or as delivering an algebraic framework for set-theoretical practice. But analysis of both interpretations seems to leave us with a dilemma. On the one hand, the ontological picture seems to fall prey of a questionable form of relativism. Under this interpretation, each set-theoretic construction is pursued through first-order descriptions which are relative to a particular set concept defined on the background of some collection of universes, i.e., different "clouds" of universes satisfying different sentences. As a consequence, what sets exist is relative to a particular set-theoretic background, and the multiverse appears as indeterminate until a particular universe is arbitrarily chosen as a starting point. This leads, according to Barton, to an unacceptable form of relativism about reference to sets, which also has severe consequences for the indeterminacy of metalogical notions such as proof and well-formed formula. On the other hand, Hamkins' view can be interpreted algebraically, as a way of telling what is possible on any structure that satisfies the *ZFC* axioms. This seems to elude the abovementioned problems with reference, but has the unwelcome drawback of leaving Benacerrafian concerns on the nature of mathematical objects and our access to them wholly untouched, and then a significant part of our mathematical practice unexplained.

One motivation underlying the various notions of multiversism featured in Barton's discussion and Friedman and Ternullo's analysis of the Hyperuniverse is to think of them as a means of reacting, in the long run, to the threat to uniqueness of truth in set theory that resulted from Cohen's forcing technique for independence results advanced in 1963. As **Giorgio Venturi** shows in his *Forcing, Multiverse and Realism*, a thorough analysis of the notion of forcing may be required to understand several philosophical aspects of contemporary conceptions of set theory. Indeed, while forcing can be made coherent with mathematical practice by saying that what we do when we extend V is just to extend what we know about it, the idea that we can force over some countable transitive model by adding sets that still lie in V seems to suggest a robustly realist idea of a kind of set-theoretic existence which is prior to, and independent of, existence in a model. One may wonder whether this notion of existence is compatible with the notion of set as shaped by the axioms of *ZFC*, and through an historical overview Venturi answers to this question in the positive. Crucial to the understanding of forcing is a sharpening of the notion of genericity, since this is a key ingredient of forcing constructions where indeed generic extensions of a countable transitive model of *ZFC* are considered. According to Venturi, who follows Mostowski's suggestions in this, a better understanding of the notion of genericity can help disentangle several philosophical issues concerning a realist view about sets, starting from a better appreciation of the notion of arbitrary set which had a pivotal role in the development of set theory. A proper study of genericity can profit from an analysis of how alternative possible bifurcation of the set-theoretic universe led to various conceptions of the multiverse. Venturi reviews some of them, ordering them according to their different attitudes towards forms of set-theoretic realism: from platonism (in Hamkins' views, cf. Hamkins 2012), through conceptualism (in Arrigoni-Friedman's Hyperuniverse, cf. Arrigoni and Friedman 2013), to semantic realism (exemplified by Woodin's conception, cf. Woodin 2001), and finally to second-order pluralism (a view attributed to Väänänen 2014). While noticing that all these views appeal to genericity without pausing on a deeper analysis of it, Venturi eventually suggests a way of exploring the notion by taking into account sets which are generic not only with respect to some particular model, but also to some multiverse structure.

Altogether, the essays in Part II explore a variety of issues stemming from realist conceptions of such a crucial domain of mathematical objects as the one set theory attempts to characterize. Despite their diverse approaches and their specific themes, all these essays display a common underlying thread: while we seem to possess an intuitive or naïve idea of what a set is, we may end up with most variegated developments of that intuitive conception, especially on the basis of the formal theories which are employed to proceed towards more rigorous treatments. This is not something affecting only the notion of set. Attempts at a proper understanding of a variety of informal notions relevant to the philosophy of mathematics are likely to lead to rather different outcomes after formalization. As we will see, the focus on this very relationship between informal notions and formal means for rigor is shared also, and even more explicitly, by the essays in Part III.

The Logic Behind Mathematics: Proof, Truth, and Formal Analysis

Although the papers in Part III of this volume cover an apparently wide variety of topics, they are all mutually and substantially connected by the effort to clarify both the philosophical consequences and significance of formal theories and formal results, and, above all, the interaction between formalization and some crucial informal notions around which much of the research in the philosophy of mathematics and logic revolves, such as the relation between proof and truth, the relations of dependence of truths on one another, the pre-theoretical intuition of the natural number structure and how it relates to our grasping of the intended model of PA, and to the formal, metatheoretical property of categoricity of PA^2. Standing at the two opposite sides of the philosophical spectrum, formal and informal notions can positively interact or conflate dramatically, but setting sharp boundaries between the two may prove to be particularly difficult. In this respect, a rather classic area of philosophical investigation concerns to what extent our pre-theoretical understanding and inquiry of informal notions philosophically inform the formal theories and our understanding and consideration thereof, or conversely to what extent our formal theories clarify or reflect our informal or even pre-theoretical intuitions. As we will see, there is a variety of ways in which truth and formal proof-theoretical settings are at the core of the essays in this last part of the volume.

Truth and Formal Theories

Throughout the papers in this Part, the connection between truth and proof is provided by formal analysis, and still the way in which these papers explore the relations between them is rather articulate. In the first two essays, the interaction between truth and proof is directly explored via strictly formal means. The effect of this approach is that the formal analysis of the truth of the Gödel sentence $\mathcal{G}$, and of the equi-interpretability of some formal theories of truth and certain set-theoretic principles helps in precisifying and rigorizing some underlying informal notions and intuitions, such as, respectively, the meaning of $\mathcal{G}$, the intertwinement of the notions of set and truth.

In the literature, there are two main schools of interpretation of the undecidable Gödel sentence $\mathcal{G}$. According to the metatheoretical meaning (see e.g., Nagel and Newman 1958), the Gödel sentence stands for a self-referential proposition claiming unprovability of itself. In order to establish the truth of $\mathcal{G}$, one has to go look into the intended model $\mathcal{N}$, and the truth of each individual instance of $\mathcal{G}$ follows immediately. On the other hand, according to a second interpretation that attributes $\mathcal{G}$ a plainly arithmetical meaning, it is the truth of the instances of $\mathcal{G}$ that secures the truth of the Gödel sentence itself. **Mario Piazza** and **Gabriele Pulcini**'s *What's so Special About the Gödel Sentence $\mathcal{G}$?* aims at providing a further argument against the metatheoretical view and in favor of the arithmetical one. Their argument is based on Dummett (1963), and proceeds by claiming that the best way to make sense of Dummett's position is to consider that, when we prove the truth of the instances of $\mathcal{G}$, on the basis of which the truth of $\mathcal{G}$ itself is established, we have to consider generic instances of $\mathcal{G}$: namely, instances of the form $\neg Prf(n, \mathcal{G})$, where n is a generic natural number—in a slightly different parlance, one might say that n is an arbitrary natural number. This kind of proof, namely the proof of the truth of $\mathcal{G}$ carried out from the proof of the truth of its generic instances, which is also envisioned in Wright (1995), is referred to as prototypical, as in Herbrand (1931). This shift in perspective allows the authors to claim that the controversy over the epistemological priority between $\mathcal{G}$ and its numerical instances, which is the core of the metatheoretical view, endures only because the problem is ultimately ill-posed. In particular, there is no way to formally prove or disprove $\mathcal{G}$ on the basis of mathematical induction, because this would require such a proof or disproof to be carried out in PA, which cannot be the case on pain of inconsistency. Consequently, Piazza and Pulcini argue, the resort to prototype reasoning is actually unavoidable in order to achieve the truth of the Gödelian sentence. The relation of epistemic priority between $\mathcal{G}$ and its numerical instances can then be revised to the effect that, by prototypical proof, there is no need to appeal to the truth of $\mathcal{G}$ in order to recognize the truth of its individual instances.

Piazza and Pulcini argue for a sharp distinction between a substantially model-theoretical view and a proof-theoretic view in the epistemic considerations concerning the relation between the truth of the Gödel sentence and the truth of its numerical instances in PA. In contrast to this, **Carlo Nicolai**'s *More on the Systems of Truth and Predicative Comprehension*, utilizes a somewhat opposite approach based on the consideration of the equi-interpretability of typed truth theory and set-theoretical predicative comprehension in order to establish a general logico-mathematical result about the interrelation of these latter formal settings. The formal results concerning the interaction between typed truth theory and predicative set existence axioms have had a rather ample echo in several fields of research, from the foundations of mathematics, to the solutions of the semantic paradoxes, and also to a possible reduction of the ontological commitment to sets to a 'lighter' ideological commitment to notions such as truth. Nevertheless, the results available in the literature are somehow limited to those theories that take PA as the base theory. Nicolai's paper investigates a generalization of these results by treating truth and set-theoretical predicative comprehension as operations on arbitrary base theories satisfying some minimal requirements, namely being recursively enumerable. Nicolai defines three main operations: $\mathsf{T}\,[\cdot]$ results in a Tarskian truth theory; $\mathsf{Tp}\,[\cdot]$ in a typed theory of truth simulating positive inductive definitions; $\mathsf{PC}\,[\cdot]$ adds predicative comprehension to the base theory. In order to study $\mathsf{PC}\,[\cdot]$ and relate it in full generality to the truth theories also studied, a variant of it, $\mathsf{PCS}\,[\cdot]$, has to be taken into account: it applies to theories axiomatized by schemata in which schematic variables are replaced by second-order variables. Modulo mutual interpretability, the extension of arbitrary recursively enumerable base theories via these three operations yields equivalent results. By this general result, Nicolai shows, among other things, how set existence principles and principles governing primitive predicates for truth or satisfaction are deeply intertwined and that a general criterion of theory choice should consider them as interdependent.

Informal Notions and Formal Analysis

The last three essays of this Part and of this volume offer yet further perspectives on the general theme of the relation between truth and proof. Here, the interaction between the latter is exploited in the effort to clarify the relation between some of our informal intuitions and notions and the way in which these are systematized in formal systems. On the one hand, we find an assessment of different logical systems trying to regiment the natural thought that truths (mathematical as well as non-mathematical) stand to each other in certain relations of dependance, and that some truths are such in virtue of other more basic ones, which ground them. On the other, we see the effort of clarifying the relation between some genuinely mathematical informal intuitions, such as the intuition of the natural number structure, and model-theoretical and metatheoretical notions, such as the intended model of PA and the categoricity of PA^2. This kind of investigations may also be able to throw light on the philosophical upshot of the interaction between a somehow more informal notion of truth (as involved in the reference to the standard model of arithmetic) and the formalization of certain mathematical frameworks, thus testifying that informal intuitions may not be completely and exhaustively captured by formal settings.

In the past decade, an entirely new field of research has opened, and its implications have fast and vastly broadened. Though first analyzed systematically in the work of Bernard Bolzano, the notion of *grounding* has recently caught massive attention and worldwide interest in many different philosophical areas such as metaphysics, logic, and philosophy of mathematics. In a nutshell, the notion of grounding should capture a certain relation of priority that holds between truths or facts, and is usually signalled in the natural language by expressive devices such as 'because' or 'in virtue of,' as in 'The ball is colored because it is blue' or 'Something is the case in virtue of something else being the case.' The logical properties of this informal notion have been extensively investigated in order to formally capture both the pre-theoretical intuitions underlying it and their metaphysical consequences (cf. Correia and Schnieder 2012). On the face of this ongoing debate, grounding is either defended as both philosophically and formally substantial, or questioned as problematic (cf. Bliss and Trogdon 2014). **Francesca Poggiolesi**, in *A Critical Overview of the Most Recent Logics of Grounding*, aims at a twofold result: on the one hand, to present in a clear and faithful way two of the most recent contributions to the logic of grounding, namely Correia (2013) and Fine (2012); on the other hand, to question the formal principles describing the notion of grounding proposed by these logics. As mentioned, the notion of grounding is rather complex and has been examined from several, different perspectives, e.g., metaphysical, historical, and logical. Since much of the formal work that has been carried out in recent years is mostly interested in the logical properties of such a notion, in order to argue for her conclusion, Poggiolesi tackles grounding from a proof-theoretical point of view. According to this perspective, grounding is a proof-theoretic relation that reveals ontological hierarchies of truths. As such, though, it has to comply with many (if not all) of the properties that have been put forward for the standard calculus of natural deduction. Nevertheless, under the assumption of the proof-theoretical nature of grounding, Poggiolesi shows that this is not the case, especially with respect to negation, disjunction, and the metalogical properties of associativity and commutativity of the conjunction and the disjunction. On this basis, Poggiolesi argues that some of the formal principles that should capture the notion of grounding in Correia's and Fine's logics need to be changed and improved.

Informal notions have always played a rather substantial part in philosophical investigations: the philosophy of mathematics is no exception. In the debate on the status of formal theories of arithmetic, our informal understanding of this branch of mathematics seems to play a role as to why the intended model of the corresponding formal theories is salient with respect to nonstandard models. So, a natural question arises: what makes the intended structure of natural numbers the standard model of arithmetic? Is there any way we can explain the emergence of $\mathbb{N}$ over nonstandard interpretations? **Massimiliano Carrara**, **Enrico Martino**, and **Matteo Plebani**'s *Computability, Finiteness and the Standard Model of Arithmetic* addresses the question of how we manage to single out the natural number structure as the intended interpretation of our arithmetical language. According to Horsten's (2012) computational structuralism, the reference of our arithmetical vocabulary to $\mathbb{N}$ is determined by our knowledge of some principles of arithmetic, like those axiomatized in PA, paired with a pre-theoretical computational capability, namely a pre-theoretical ability to compute sums. Carrara, Martino, and Plebani take issue with such a view and submit an alternative answer to the question concerning the salience of the standard model of arithmetic. According to the authors, both our understanding of the axioms of PA and of how to compute sums correctly rest on something more fundamental, namely our ability to generate the relevant syntactical entities that constitute a formal theory like PA and are the basis on which the addition algorithm works. This, in turn, rests on our ability to grasp a primitive notion of finiteness. It is the intuition of this latter pre-theoretical, absolute notion of finiteness that allows the singling out of the structure of natural numbers.

While Carrara, Martino and Plebani focus on a specific proposal, i.e., Horsten's computational structuralism, concerning the salience of the intended model of PA, by rejecting it and advancing a further suggestion based on the pre-theoretical notion of absolute finiteness, **Samantha Pollock**'s *The Significance of a Categoricity Theorem for Formal Theories and Informal Beliefs* scrutinizes the role that categoricity plays in the interaction between our beliefs about informal mathematical theories (e.g., arithmetic) and the properties enjoyed by formal mathematical systems (e.g., PA^2). By offering a characterization of the requirements a theory should satisfy in order to be legitimately considered as either formal or informal, a pattern of informal notions mirrored by formal properties is recognized: for instance, we informally require that an informal mathematical theory is about a unique model or structure (e.g., the natural number structure as for arithmetic) or has an intended interpretation. These kind of informal properties are invoked in discussions on the significance of categoricity for formal mathematical theories. On this view, categoricity shows to be a two-faced property, having two kinds of philosophical significance. On the one hand, it has formal significance when it is invoked with respect to formalization: "An argument pertains to the formal significance of categoricity if it takes some informal beliefs about" an informal mathematical theory "T_I, and assesses the extent to which being categorical makes a formalization T_F of T_I adequate (i.e., faithful) with respect to those beliefs." On the other hand, it may be invoked with respect to informal mathematical theories: "An argument pertains to the informal significance of categoricity if it takes a particular formalization T_F of T_I as adequate (i.e., faithful), and assesses the extent to which its being categorical (or not) is instructive with respect to what we should informally believe about T_I." Potential consequences of this distinction arise. In particular, a potential source of circularity in Shapiro's *ante rem* mathematical structuralism (see Shapiro 1991) arises out of what appear to be arguments for both kinds of philosophical significance with respect to categoricity, thus showing that implicit claims surrounding the significance of categoricity can lead to philosophical missteps without due caution.

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In Memoriam of Aldo Antonelli

Andrew Arana and Curtis Franks

These days people are quick to take positions, “hot takes.” *ZFC* is the right foundation of mathematics, first-order logic is the true logic, the nature of mathematical knowledge is intuitionist, the axiom of projective determinism is true, nominalism is true, and so on. These are bold statements, and they draw attention. The result is the literature that moves quickly and as a result yields little change, little persuasion, little clarification, and little wisdom. It comes to seem dogmatic: one’s reputation is connected with one’s intellectual position, and thus changing your mind comes with the loss of professional status.

Aldo was not like this. Aldo’s work was about exploring new possibilities, alternatives to views that could otherwise calcify into mere dogmas. One can think of them as playing a role in an intellectual infrastructure to which spaces are regularly wrongly closed off. “that approach won’t work, for such and such a reason”: Aldo would provide technical results that opened those approaches up again. Whether you take those approaches, that’s up to you. But it’s not their impossibility that should stop you.

Mathematics has a reputation in the academy of being dry and lifeless, and among mathematical topics, logic has this reputation more than any other. Aldo didn’t see it this way. “I will take beauty over truth any day,” he would say, and while perhaps jarring at first, any seasoned mathematician will recognize the insight. Aldo, who could do just about anything, focused on logic because that found it to be incomparably beautiful. It shaped the way he taught and also the way he advised students. Discussing an amazingly talented undergraduate student with interests in logic as well as existentialism and post-Kantian themes, Aldo’s idea of

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how to encourage her to look more into logic was to say "Just tell her that it is so beautiful, that she will love it." His sense and love of beauty fueled his distinct sense of what questions were worth asking, what questions worth pursuing. Aldo spent hours every day just working through recent papers and central texts in algebra, set theory, topology, geometry, and analysis. But he never wrote about these things. Aldo knew, not everything in modern mathematics, but close to everything that was beautiful.

We'll continue with a brief discussion of Aldo's early work. He worked on the foundations of defeasible consequence and non-monotonic logics, of Quine's New Foundations set theory, and of non-well-founded set theory. Rather than try to summarize the results, we can see this work as expressing the order of Aldo's approach to problems. One might say that there were some technical gimmicks to work out, and he had the relevant tools to discover and present these things. But this is wrong. Aldo was interested in the results that were the most beautiful; the ones that unpacked the most hidden connections, the ones that made us rethink the greatest number of our inhibiting preconceptions. Because he didn't care much about, and possibly didn't even understand, ideas about which set theories were correct, he was able to feel his way to the mathematical relationships that disclosed the greatest number of such insights.

Aldo applied this method with particular focus to logicism and in particular to the thesis that arithmetic is logic. Frege's logicist program had the following two stages: (1) to define numbers as extensions of concepts, and (2) to derive logically the theorems of arithmetic from that definition. That these two stages cannot both be carried out is more or less consensus today. Where there is consensus, though, there is need for light: for what possibilities are being closed out by the consensus? We want to talk about three projects of Aldo that opened such possibilities for logicism.

We'll begin with "Frege's New Science," written with Robert May. Aldo broke the question of whether Frege's logic could carry out metatheory into two parts: (1) can metalogical questions about Frege's logic be posed in Frege's logic itself? and (2) is metatheory necessarily model theory, in which one varies the meanings of propositions in order to prove for instance independence results? Aldo and Robert answered "no" to (2), in light of Frege's insistance that one cannot reinterpret the meanings (references) of nonlogical terms of axioms, since axioms express thoughts. They argue that Frege saw, if not particularly clearly, a way to develop metatheory in which one replaces the reinterpretability of meanings with a kind of permutability of nonlogical vocabulary on which no meanings are changed. While the question of what vocabulary is nonlogical arises, Aldo and Robert sketch an argument (at least nearly) available to the Fregean that this question too can be handled by a permutability argument. Thus this article opens space for a new approach to understanding Frege's metatheory by pursuing a technical development.

Next, let's turn to "Frege's Other Program," also written with Robert May. Here Aldo and Robert explore a different possibility for reckoning the two logicist stages of defining numbers as extensions of concepts, and of deriving logically the theorems of arithmetic from that definition. As Aldo and Robert put it, one can attempt "to show in a nonlogical theory of extensions, where numbers are concepts, not objects, that Peano Arithmetic can be derived." In doing so they identify a non-logicist but still broadly Fregean program for deriving arithmetic. This program clarifies the causes of the contradiction entailed by Basic Law V, providing a new counterexample to Hume's Principle. Here again new spaces are opened, by investigating an extensional theory of arithmetic without supposing Hume's Principle. One can then consider to what extent such a theory could be judged a vindication of logicism.

Next, we'll turn to Aldo's 2010 article "The Nature and Purpose of Numbers," notable not least because it appeared in one of American philosophy's top journals, the *Journal of Philosophy*. This article investigates a possible version of Fregean logicism, one that differs from other developments in that it does not reduce arithmetic to set theory. Aldo takes cardinal properties of the natural numbers as the starting point, and derives structural properties of the natural numbers from them, rather than the other way around as is done in typical set-theoretic reductions of arithmetic. Aldo's idea is that "in keeping with the broadest and most general construal of logicism, cardinality notions… deal directly with properties and relations of concepts—rather than matters of existence of objects such as numbers—[and thus] cardinality notions properly can be regarded as having a logical character." They are logical notions, Aldo argues, because relations of concepts are quantifiers: more precisely, they are, Aldo argues, generalized quantifiers. The ordinary existential and universal quantifiers can be thought of as relations of concepts: the existential quantifier as the collection of all nonempty subsets of the domain of quantification, and the universal quantifier as the collection of all subsets of the domain that contain the domain as their only member. But these are not the only two relations that yield quantifiers, on the theory of generalized quantifiers, and in particular one can consider the Frege quantifier, which holds between two concepts F and G when there are no more Fs than Gs. In fact, these are all first-order quantifiers, Aldo argues. He then shows how the Frege quantifier can be taken as logically basic, and shows how one can derive the basic features of the natural numbers from such a logic.

We'd like to note in particular Aldo's way of characterizing his accomplishment in this article. "Accordingly, we take the logicist claim that cardinality is a logical notion at face value, and rather than arguing for it (perhaps by providing a reduction to some other principle), we set out to explore its consequences by introducing cardinality, in the form of the Frege quantifier, as the main building block in the language of arithmetic."

This passage illustrates beautifully the idea that Aldo's approach to problems was prescind from particular theoretical stances, and to explore the consequences of these stances. With a clearer understanding of these consequences, of their fruits, one can better evaluate the costs and benefits of particular positions. The job of the logician is to explore these consequences. And Aldo was a logician.

In this 2010 article, Aldo also noted how the Frege quantifier, a first-order quantifier, can be given a generalized Henkin interpretation. In Henkin's work, one restricts the range of quantification to just a subset of the power set of the full domain. Models for second-order logic can then be specified by giving both the domain and a universe of relations over the domain. Aldo's novel insight was that such an interpretation can also be given for first-order quantifiers. Within the context of Fregean logicism, this permits one to prescind from concerns about whether second-order logic is logic (another one of those dogmatic debates).

His 2013 article in the *Review of Symbolic Logic*, "On the general interpretation of first-order quantifiers," expanded on this novel insight. 60 years or so after Henkin's groundbreaking work on generalized models, Aldo observed what no one else ever noticed, namely, that the notion of a generalized model can be formulated already for first-order languages. The irony is sharp: The "first-order case" of the Henkin construction becomes an extension (not a restriction) of the familiar second-order case, where the notion of models given by filters over the full power set construction is more intuitive. This is, in our opinion, Aldo's deepest work.

In an article entitled "Life on the Range: Quine's Thesis and Semantic Indeterminacy," published this summer, Aldo pursued the consequences of this technical development for the evaluation of Quine's dictum that to be is to be the value of a bound variable. As Aldo notes, this dictum flows from Quine's view that second-order logic is "set theory in sheep's clothing." Since second-order logic on Quine's view has ontological commitments, it is not really logic. Aldo observes that his work on generalized models puts pressure on Quine's views. Since the first-order quantifiers can be interpreted to be extensions of second-order quantifiers, the ontological commitments of second-order logic are also ontological commitments of first-order quantifiers. In the closing sentence of this article, Aldo writes that "this last realization can contribute to the establishment of second-order logic on the same safe footing as first-order logic."

In closing, it seems to us that most professional philosophers spend more time advancing their own research programs than they spend learning. This strikes us as completely unreasonable, and we are pretty sure that our attitude derives from Aldo's influence. We ask ourselves: Why would I, or anyone really, care more about what I have to say than about what some 20 or so brilliant historical figures have already said? Do I love logic, math, and philosophy, or do I love professional credits? Everyone in our world initially loved the former, and it is a disgrace, Aldo taught, to abandon this idea. And, he taught, if you persevere in your love for the most beautiful ideas in mathematics, philosophy, and logic, your own contributions will trickle in at the right time. Those ideas will not come close to being the most interesting things you have to talk about. But they will not only be true, they will be beautiful.