UNITEXT – La Matematica per il 3+2

Volume 99

Editor-in-chief

A. Quarteroni

Series editors

L. Ambrosio

P. Biscari

C. Ciliberto

M. Ledoux

W.J. Runggaldier

More information about this series at <http://www.springer.com/series/5418>

Sandro Salsa

Partial Differential Equations in Action

From Modelling to Theory

Third Edition

Springer

Sandro Salsa
Dipartimento di Matematica
Politecnico di Milano
Milano, Italy

ISSN 2038-5722
UNITEXT – La Matematica per il 3+2
ISBN 978-3-319-31237-8
DOI 10.1007/978-3-319-31238-5

ISSN 2038-5757 (electronic)
ISBN 978-3-319-31238-5 (eBook)

Library of Congress Control Number: 2016932390

© Springer International Publishing Switzerland 2016

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made.

Cover illustration: Simona Colombo, Giochi di Grafica, Milano, Italy
Typesetting with $\LaTeX$: PTP-Berlin, Protago $\TeX$ -Production GmbH, Germany (www.ptp-berlin.de)

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG Switzerland

To Anna, my wife

Preface

This book is designed as an advanced undergraduate or a first-year graduate course for students from various disciplines like applied mathematics, physics, engineering. It has evolved while teaching courses on partial differential equations (PDEs) during the last few years at the Politecnico di Milano.

The main purpose of these courses was twofold: on the one hand, to train the students to appreciate the interplay between theory and modelling in problems arising in the applied sciences, and on the other hand to give them a solid theoretical background for numerical methods, such as finite elements.

Accordingly, this textbook is divided into two parts, that we briefly describe below, writing in *italics* the relevant differences with the first edition, the second one being pretty similar.

The **first part**, Chaps. 2 to 5, has a rather elementary character with the goal of developing and studying basic problems from the macro-areas of diffusion, propagation and transport, waves and vibrations. I have tried to emphasize, whenever possible, ideas and connections with concrete aspects, in order to provide intuition and feeling for the subject.

For this part, a knowledge of advanced calculus and ordinary differential equations is required. Also, the repeated use of the method of separation of variables assumes some basic results from the theory of Fourier series, which are summarized in Appendix A.

Chapter 2 starts with the heat equation and some of its variants in which transport and reaction terms are incorporated. In addition to the classical topics, I emphasized the connections with simple stochastic processes, such as random walks and Brownian motion. This requires the knowledge of some elementary probability. It is my belief that it is worthwhile presenting this topic as early as possible, even at the price of giving up to a little bit of rigor in the presentation. An application to financial mathematics shows the interaction between probabilistic and deterministic modelling. The last two sections are devoted to two simple non linear models from flow in porous medium and population dynamics.

Chapter 3 mainly treats the Laplace/Poisson equation. The main properties of harmonic functions are presented once more emphasizing the probabilistic motivations. *I have included Perron's method of sub/super solution*, due to its renewed importance as a solution technique for fully non linear equations. The second part of this chapter deals with representation formulas in terms of potentials. In particular, the basic properties of the single and double layer potentials are presented.

Chapter 4 is devoted to first order equations and in particular to first order scalar conservation laws. The methods of characteristics and the notions of shock and rarefaction waves are introduced through a simple model from traffic dynamics. *An application to sedimentation theory illustrates the method for non convex/concave flux function*. In the last part, the method of characteristics is extended to quasilinear and fully nonlinear equations in two variables.

In Chap. 5 the fundamental aspects of waves propagation are examined, leading to the classical formulas of d'Alembert, Kirchhoff and Poisson. *A simple model of Acoustic Thermography serves as an application of Huygens principle*. In the final section, the classical model for surface waves in deep water illustrates the phenomenon of dispersion, with the help of the method of stationary phase.

The **second part** includes the two *new Chaps. 9 and 11*. In Chaps. 6 to 10 we develop Hilbert spaces methods for the variational formulation and the analysis of mainly linear boundary and initial-boundary value problems. Given the abstract nature of these chapters, I have made an effort to provide intuition and motivation about the various concepts and results, sometimes running the risk of appearing a bit wordy. The understanding of these topics requires some basic knowledge of Lebesgue measure and integration, summarized in Appendix B.

Chapter 6 contains the tools from functional analysis in Hilbert spaces, necessary for a correct variational formulation of the most common boundary value problems. The main theme is the solvability of abstract variational problems, leading to the Lax-Milgram theorem and Fredholm's alternative. Emphasis is given to the issues of compactness and weak convergence. *Section 6.10 is devoted to the fixed point theorems of Banach and of Schauder and Leray-Schauder*.

Chapter 7 is divided into two parts. The first one is a brief introduction to the theory of distributions (or generalized functions) of L. Schwartz. In the second one, the most used Sobolev spaces and their basic properties are discussed.

Chapter 8 is devoted to the variational formulation of linear elliptic boundary value problems and their solvability. The development starts with Poisson's equation and ends with general second order equations in divergence form.

In Chap. 9 I have gathered a number of applications of the variational theory of elliptic equations, in particular *to elastostatics and to the stationary Navier-Stokes equations*. Also, an application to a simple control problem is discussed.

The issue in Chap. 10, which has been *almost completely remodeled*, is the variational formulation of evolution problems, in particular of initial-boundary value problems for second order parabolic operators in divergence form and for the wave equation.

Chapter 11 contains a brief introduction to the basic concepts of the *theory of systems of first order conservation laws, in one spatial dimension*. In particular we extend from the scalar case of Chap 4, the notions of characteristics, shocks, rarefaction waves, contact discontinuity and entropy condition. The main focus is the solution of the Riemann problem.

At the end of each chapter, a number of exercises is included. Some of them can be solved by a routine application of the theory or of the methods developed in the text. Other problems are intended as a completion of some arguments or proofs in the text. Also, there are problems in which the student is required to be more autonomous. The most demanding problems are supplied with answers or hints.

Other (completely solved) exercises can be found in [17], the natural companion of this book by *S. Salsa, G. Verzini*, Springer 2015.

The order of presentation of the material is clearly a consequence of my ... prejudices. However, the exposition is flexible enough to allow substantial changes without compromising the comprehension and to facilitate a selection of topics for a one or two semester course.

In the first part, the chapters are, in practice, mutually independent, with the exception of Subsection 3.3.1 and Sect. 3.4, which presume the knowledge of Sect. 2.6.

In the second part, more attention has to be paid to the order of the arguments. The material in Sects. 6.1–6.9 and in Sect. 7.1–7.4 and 7.7–7.10 is necessary for understanding the topics in Chap. 8–10. Moreover, Chap. 9 requires also Sect. 6.10, while to cover Chap. 10, also concepts and results from Sect. 7.11 are needed.

Finally, Chap. 11 uses Subsections 4.4.2, 4.4.3 and 4.6.1.

Acknowledgments. While writing this book, during the first edition, I benefited from comments and suggestions of many colleagues and students.

Among my colleagues, I express my gratitude to Luca Dedé, Fausto Ferrari, Carlo Pagani, Kevin Payne, Alfio Quarteroni, Fausto Saleri, Carlo Sgarra, Alessandro Veneziani, Gianmaria A. Verzini and, in particular to Cristina Cerutti, Leonede De Michele and Peter Laurence.

Among the students who have sat through my course on PDEs, I would like to thank Luca Bertagna, Michele Coti-Zelati, Alessandro Conca, Alessio Fumagalli, Loredana Gaudio, Matteo Lesinigo, Andrea Manzoni and Lorenzo Tamellini.

For the last two editions, I am particularly indebted to Leonede de Michele, Ugo Gianazza and Gianmaria Verzini for their interest, criticism and contribution. Many thanks go to Michele Di Cristo, Giovanni Molica-Bisci, Nicola Parolini, Attilio Rao and Francesco Tulone for their comments and the time we spent in precious (for me) discussions. Finally, I like to express my appreciation to Francesca Bonadei and Francesca Ferrari of Springer Italia, for their constant collaboration and support.

Contents

1	Introduction	1
1.1	Mathematical Modelling	1
1.2	Partial Differential Equations	2
1.3	Well Posed Problems	5
1.4	Basic Notations and Facts	7
1.5	Smooth and Lipschitz Domains	12
1.6	Integration by Parts Formulas	15
2	Diffusion	17
2.1	The Diffusion Equation	17
2.1.1	Introduction	17
2.1.2	The conduction of heat	18
2.1.3	Well posed problems ($n = 1$)	20
2.1.4	A solution by separation of variables	23
2.1.5	Problems in dimension $n > 1$	32
2.2	Uniqueness and Maximum Principles	34
2.2.1	Integral method	34
2.2.2	Maximum principles	36
2.3	The Fundamental Solution	39
2.3.1	Invariant transformations	39
2.3.2	The fundamental solution ($n = 1$)	41
2.3.3	The Dirac distribution	43
2.3.4	The fundamental solution ($n > 1$)	47
2.4	Symmetric Random Walk ($n = 1$)	48
2.4.1	Preliminary computations	49
2.4.2	The limit transition probability	52
2.4.3	From random walk to Brownian motion	54
2.5	Diffusion, Drift and Reaction	58
2.5.1	Random walk with drift	58
2.5.2	Pollution in a channel	60
2.5.3	Random walk with drift and reaction	63

2.5.4	Critical dimension in a simple population dynamics	64
2.6	Multidimensional Random Walk	66
2.6.1	The symmetric case	66
2.6.2	Walks with drift and reaction	70
2.7	An Example of Reaction–Diffusion in Dimension $n = 3$	71
2.8	The Global Cauchy Problem ($n = 1$)	76
2.8.1	The homogeneous case	76
2.8.2	Existence of a solution	78
2.8.3	The nonhomogeneous case. Duhamel’s method	79
2.8.4	Global maximum principles and uniqueness.	82
2.8.5	The proof of the existence theorem 2.12	85
2.9	An Application to Finance	88
2.9.1	European options	88
2.9.2	An evolution model for the price S	89
2.9.3	The Black-Scholes equation	91
2.9.4	The solutions	95
2.9.5	Hedging and self-financing strategy	100
2.10	Some Nonlinear Aspects	102
2.10.1	Nonlinear diffusion. The porous medium equation	102
2.10.2	Nonlinear reaction. Fischer’s equation	105
	Problems	109
3	The Laplace Equation	115
3.1	Introduction	115
3.2	Well Posed Problems. Uniqueness.	116
3.3	Harmonic Functions	118
3.3.1	Discrete harmonic functions	118
3.3.2	Mean value properties	122
3.3.3	Maximum principles	124
3.3.4	The Hopf principle	126
3.3.5	The Dirichlet problem in a disc. Poisson’s formula	127
3.3.6	Harnack’s inequality and Liouville’s theorem	131
3.3.7	Analyticity of harmonic functions	133
3.4	A probabilistic solution of the Dirichlet problem	135
3.5	Sub/Superharmonic Functions. The Perron Method	140
3.5.1	Sub/superharmonic functions	140
3.5.2	The method	142
3.5.3	Boundary behavior	143
3.6	Fundamental Solution and Newtonian Potential	147
3.6.1	The fundamental solution	147
3.6.2	The Newtonian potential	148
3.6.3	A divergence-curl system. Helmholtz decomposition formula	151
3.7	The Green Function	155
3.7.1	An integral identity	155

3.7.2	Green's function	157
3.7.3	Green's representation formula	160
3.7.4	The Neumann function	161
3.8	Uniqueness in Unbounded Domains	163
3.8.1	Exterior problems	163
3.9	Surface Potentials	166
3.9.1	The double and single layer potentials	166
3.9.2	The integral equations of potential theory	171
	Problems	174
4	Scalar Conservation Laws and First Order Equations	179
4.1	Introduction	179
4.2	Linear Transport Equation	180
4.2.1	Pollution in a channel	180
4.2.2	Distributed source	182
4.2.3	Extinction and localized source	183
4.2.4	Inflow and outflow characteristics. A stability estimate	185
4.3	Traffic Dynamics	187
4.3.1	A macroscopic model	187
4.3.2	The method of characteristics	189
4.3.3	The green light problem	191
4.3.4	Traffic jam ahead	196
4.4	Weak (or Integral) Solutions	199
4.4.1	The method of characteristics revisited	199
4.4.2	Definition of weak solution	202
4.4.3	Piecewise smooth functions and the Rankine-Hugoniot condition	205
4.5	An Entropy Condition	209
4.6	The Riemann problem	212
4.6.1	Convex/concave flux function	212
4.6.2	Vanishing viscosity method	214
4.6.3	The viscous Burgers equation	218
4.6.4	Flux function with inflection points	220
4.7	An Application to a Sedimentation Problem	224
4.8	The Method of Characteristics for Quasilinear Equations	230
4.8.1	Characteristics	230
4.8.2	The Cauchy problem	232
4.8.3	Lagrange method of first integrals	239
4.8.4	Underground flow	241
4.9	General First Order Equations	244
4.9.1	Characteristic strips	244
4.9.2	The Cauchy Problem	246
	Problems	251

5	Waves and Vibrations	259
5.1	General Concepts	259
5.1.1	Types of waves	259
5.1.2	Group velocity and dispersion relation	261
5.2	Transversal Waves in a String	264
5.2.1	The model	264
5.2.2	Energy	266
5.3	The One-dimensional Wave Equation	267
5.3.1	Initial and boundary conditions	267
5.3.2	Separation of variables	269
5.4	The d'Alembert Formula	275
5.4.1	The homogeneous equation	275
5.4.2	Generalized solutions and propagation of singularities	279
5.4.3	The fundamental solution	282
5.4.4	Nonhomogeneous equation. Duhamel's method	285
5.4.5	Dissipation and dispersion	286
5.5	Second Order Linear Equations	288
5.5.1	Classification	288
5.5.2	Characteristics and canonical form	291
5.6	The Multi-dimensional Wave Equation ($n > 1$)	296
5.6.1	Special solutions	296
5.6.2	Well posed problems. Uniqueness	298
5.7	Two Classical Models	302
5.7.1	Small vibrations of an elastic membrane	302
5.7.2	Small amplitude sound waves	306
5.8	The Global Cauchy Problem	310
5.8.1	Fundamental solution ($n = 3$) and strong Huygens' principle	310
5.8.2	The Kirchhoff formula	313
5.8.3	The Cauchy problem in dimension 2	316
5.9	The Cauchy Problem with Distributed Sources	318
5.9.1	Retarded potentials ($n = 3$)	318
5.9.2	Radiation from a moving point source	320
5.10	An Application to Thermoacoustic Tomography	324
5.11	Linear Water Waves	328
5.11.1	A model for surface waves	328
5.11.2	Dimensionless formulation and linearization	332
5.11.3	Deep water waves	334
5.11.4	Interpretation of the solution	336
5.11.5	Asymptotic behavior	338
5.11.6	The method of stationary phase	340
	Problems	342

6	Elements of Functional Analysis	347
6.1	Motivations	347
6.2	Norms and Banach Spaces	353
6.3	Hilbert Spaces	358
6.4	Projections and Bases	363
6.4.1	Projections	363
6.4.2	Bases	367
6.5	Linear Operators and Duality	373
6.5.1	Linear operators	373
6.5.2	Functionals and dual space	377
6.5.3	The adjoint of a bounded operator	379
6.6	Abstract Variational Problems	382
6.6.1	Bilinear forms and the Lax-Milgram Theorem	382
6.6.2	Minimization of quadratic functionals	387
6.6.3	Approximation and Galerkin method	388
6.7	Compactness and Weak Convergence	391
6.7.1	Compactness	391
6.7.2	Compactness in $C(\overline{\Omega})$ and in $L^p(\Omega)$	392
6.7.3	Weak convergence and compactness	393
6.7.4	Compact operators	397
6.8	The Fredholm Alternative	399
6.8.1	Hilbert triplets	399
6.8.2	Solvability for abstract variational problems	402
6.8.3	Fredholm's alternative	405
6.9	Spectral Theory for Symmetric Bilinear Forms	407
6.9.1	Spectrum of a matrix	407
6.9.2	Separation of variables revisited	407
6.9.3	Spectrum of a compact self-adjoint operator	408
6.9.4	Application to abstract variational problems	411
6.10	Fixed Points Theorems	416
6.10.1	The Contraction Mapping Theorem	417
6.10.2	The Schauder Theorem	418
6.10.3	The Leray-Schauder Theorem	420
	Problems	421
7	Distributions and Sobolev Spaces	427
7.1	Distributions. Preliminary Ideas	427
7.2	Test Functions and Mollifiers	429
7.3	Distributions	433
7.4	Calculus	438
7.4.1	The derivative in the sense of distributions	438
7.4.2	Gradient, divergence, Laplacian	440
7.5	Operations with Distributions	443
7.5.1	Multiplication. Leibniz rule	443
7.5.2	Composition	444

7.5.3	Division	448
7.5.4	Convolution	449
7.5.5	Tensor or direct product	451
7.6	Tempered Distributions and Fourier Transform	454
7.6.1	Tempered distributions	454
7.6.2	Fourier transform in $\mathcal{S}'$	457
7.6.3	Fourier transform in L^2	460
7.7	Sobolev Spaces	461
7.7.1	An abstract construction	461
7.7.2	The space $H^1(\Omega)$	462
7.7.3	The space $H_0^1(\Omega)$	466
7.7.4	The dual of $H_0^1(\Omega)$	467
7.7.5	The spaces $H^m(\Omega)$, $m > 1$	470
7.7.6	Calculus rules	471
7.7.7	Fourier transform and Sobolev spaces	473
7.8	Approximations by Smooth Functions and Extensions	474
7.8.1	Local approximations	474
7.8.2	Extensions and global approximations	475
7.9	Traces	479
7.9.1	Traces of functions in $H^1(\Omega)$	479
7.9.2	Traces of functions in $H^m(\Omega)$	483
7.9.3	Trace spaces	484
7.10	Compactness and Embeddings	487
7.10.1	Rellich's theorem	487
7.10.2	Poincaré's inequalities	488
7.10.3	Sobolev inequality in $\mathbb{R}^n$	490
7.10.4	Bounded domains	492
7.11	Spaces Involving Time	494
7.11.1	Functions with values into Hilbert spaces	494
7.11.2	Sobolev spaces involving time	497
	Problems	499
8	Variational Formulation of Elliptic Problems	505
8.1	Elliptic Equations	505
8.2	Notions of Solutions	507
8.3	Problems for the Poisson Equation	509
8.3.1	Dirichlet problem	509
8.3.2	Neumann, Robin and mixed problems	512
8.3.3	Eigenvalues and eigenfunctions of the Laplace operator	517
8.3.4	An asymptotic stability result	519
8.4	General Equations in Divergence Form	521
8.4.1	Basic assumptions	521
8.4.2	Dirichlet problem	522
8.4.3	Neumann problem	527
8.4.4	Robin and mixed problems	530

8.5	Weak Maximum Principles	531
8.6	Regularity	536
	Problems	544
9	Further Applications	551
9.1	A Monotone Iteration Scheme for Semilinear Equations	551
9.2	Equilibrium of a Plate	554
9.3	The Linear Elastostatic System	556
9.4	The Stokes System	561
9.5	The Stationary Navier Stokes Equations	566
9.5.1	Weak formulation and existence of a solution	566
9.5.2	Uniqueness	569
9.6	A Control Problem	571
9.6.1	Structure of the problem	571
9.6.2	Existence and uniqueness of an optimal pair	572
9.6.3	Lagrange multipliers and optimality conditions	574
9.6.4	An iterative algorithm	575
	Problems	576
10	Weak Formulation of Evolution Problems	581
10.1	Parabolic Equations	581
10.2	The Cauchy-Dirichlet Problem for the Heat Equation	583
10.3	Abstract Parabolic Problems	586
10.3.1	Formulation	586
10.3.2	Energy estimates. Uniqueness and stability.	589
10.3.3	The Faedo-Galerkin approximations.	591
10.3.4	Existence	592
10.4	Parabolic PDEs	593
10.4.1	Problems for the heat equation	593
10.4.2	General Equations	596
10.4.3	Regularity	598
10.5	Weak Maximum Principles	600
10.6	The Wave Equation	602
10.6.1	Hyperbolic Equations	602
10.6.2	The Cauchy-Dirichlet problem	603
10.6.3	The method of Faedo-Galerkin	605
10.6.4	Solution of the approximate problem	606
10.6.5	Energy estimates	607
10.6.6	Existence, uniqueness and stability	609
	Problems	611
11	Systems of Conservation Laws	615
11.1	Introduction	615
11.2	Linear Hyperbolic Systems	620
11.2.1	Characteristics	620

11.2.2	Classical solutions of the Cauchy problem	621
11.2.3	Homogeneous systems with constant coefficients. The Riemann problem	623
11.3	Quasilinear Conservation Laws	627
11.3.1	Characteristics and Riemann invariants	627
11.3.2	Weak (or integral) solutions and the Rankine-Hugoniot condition	630
11.4	The Riemann Problem	631
11.4.1	Rarefaction curves and waves. Genuinely nonlinear systems	633
11.4.2	Solution of the Riemann problem by a single rarefaction wave	636
11.4.3	Lax entropy condition. Shock waves and contact discontinuities	638
11.4.4	Solution of the Riemann problem by a single k -shock	640
11.4.5	The linearly degenerate case	642
11.4.6	Local solution of the Riemann problem	643
11.5	The Riemann Problem for the p -system	644
11.5.1	Shock waves	644
11.5.2	Rarefaction waves	646
11.5.3	The solution in the general case	649
	Problems	653
Appendix A. Fourier Series		657
A.1	Fourier Coefficients	657
A.2	Expansion in Fourier Series	660
Appendix B. Measures and Integrals		663
B.1	Lebesgue Measure and Integral	663
B.1.1	A counting problem	663
B.1.2	Measures and measurable functions	665
B.1.3	The Lebesgue integral	667
B.1.4	Some fundamental theorems	668
B.1.5	Probability spaces, random variables and their integrals	670
Appendix C. Identities and Formulas		673
C.1	Gradient, Divergence, Curl, Laplacian	673
C.2	Formulas	675
References		677
Index		681