Part I
Monodromy in Linear Differential Equations

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To give divergent power series an analytic meaning via summability and resurgence theory is a central issue in this three volume work. Divergent solutions of a differential equation account for the presence of singular points which in general prevent local analytic solutions to extend as single-valued functions on the punctured complex plane. For linear differential equations, the monodromy and Stokes matrices 'measure' the multivaluedness of the solutions, depending on the regularity or irregularity of the singular points. In the regular singular case, the monodromy representation gives a geometric description of the differential equation. In the irregular case, the Stokes matrices which arise from formal divergent solutions, are together with the monodromy matrices elements of the differential Galois group. This is a linear algebraic group which provides an algebraic interpretation of the differential equation : of its solvability for instance, or the existence of transcendental solutions.

The first chapter of this volume is an elementary introduction to analytic continuation, monodromy and singular points, with detailed definitions and proofs.

The second chapter is devoted to differential Galois theory, with basic facts over general differential fields followed by the analytic theory over the field of complex rational functions. With a view to the direct problem of calculating differential Galois groups, we present two important density theorems: Schlesinger's theorem, which relates the differential Galois group of differential systems with regular singularities to their monodromy, and Ramis's density theorem for irregular singularities, which describes the differential Galois group in terms of more specific invariants than the monodromy.

The third chapter gives a short overview of inverse problems, from the Riemann-Hilbert problem to differential inverse Galois problems via the Tretkoff theorem. Recent developments of these problems are mentioned, with references.

The fourth chapter is an introduction to the Riemann-Hilbert problem from Bolibrukh's point of view. To produce his famous counterexample, Bolibrukh introduced specific methods to attack the still open problem of characterizing those monodromy representations which can be realized by Fuchsian systems. We explain these methods and give some accessible proofs, once the necessary and elementary material about fiber bundles is presented.

The following notes grew out of lectures given to students of the CIMPA school with a diverse variety of backgrounds. We therefore presented the introductory parts of the subject in more detail than we normally would have in a graduate course. We decided to reproduce these notes here, hoping they will give the beginners an easier access to the more specialized parts of the three volumes.