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Lev Truskinovsky, Laboratoire de Mechanique des Solid, Palaiseau, France
e-mail: trusk@lms.polytechnique.fr

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Serge Preston

Non-commuting Variations in Mathematics and Physics

A Survey

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NOTES ON THE NONCOMMUTING VARIATIONS.

SERGE PRESTON

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1. Preface

The aim of this text is to present and study the method of so-called "**non-commuting variations (shortly, NC-variations)**" in Variational Calculus. To present this method we recall one of the basic rules of Variational Calculus - the rule defining the variations of derivatives $\frac{\partial y^\mu}{\partial x^i}$ of dynamical variables $y^\mu(x)$ (fields in the Field Theory) corresponding to a variation ξ of dynamical variables (fields): "variation of a derivative equals to the derivative of variation". In Mechanics, this rule takes the form $\delta \dot{y}^\mu = \frac{d}{dt}\delta y^\mu$. In Classical Field theory this rule takes the form

$$\delta \frac{\partial y^\mu}{\partial x^i} = \frac{\partial}{\partial x^i}\delta y^\mu .$$

This rule can be formulated as follows: "taking of variations of dynamical variables $y^i(x)$ commute with the taking of derivatives."

This rule was universally adopted in the XVIII and XIX centuries but, as early as in 1887, this rule was questioned by Vito Volterra, see [130, 131]. Studying non-holonomic mechanical systems, V.Volterra noticed that the use of the conventional rule of defining variations of derivatives does not allows us to obtain equations of motion for non-holonomic systems by variational methods. Further developments including works of L.Boltzman,[8] G.Hamel,[57], T.Levi Civita and U.Amaldi,[83] led to the conflicting points of view at the range of applicability of the conventional rule of defining variations (see a Historical Review between Chapters 1 and 2 below). Finally, the status of this, conventional, rule and its relation to the alternative rules - the use of "non-commuting variations" in Non-Holonomic Mechanics were clarified in works of J.Neimark and his coauthors in the 1950s of XX century ([104, 105]) and by A.Lurie in 1961, [88].

Later on, the non-commuting variations were used in the works of B.Vujanovich and T.Atanackovic on dynamical systems with non-conservative forces ([133, 134, 4, 5]), in Elasticity Theory, and in works of H.Kleinert, P.Fiziev and A.Pelster on the dynamics in Cartan-Riemann spaces ([35, 65]).

While studying the application of non-commuting variations in classical field theory we noticed that the usage of non-commuting rules to define variations of derivatives **is equivalent to the use of a non-trivial vertical connections** to modify the procedure of flow prolongation of variational vector fields in the space Y of the configurational bundle $\pi : Y \to X$ of a physical system to the 1-jet bundle $J^1(\pi) \to Y$ over π, [112]. This led us to the study of the geometrical structures underlying the method of non-commuting variations of derivatives in Lagrangian formalism. In particular,a natural variety of questions that arises here is: which of the basic methods of Variational Calculus - Theory of second variations, Hamiltonian systems and Legendre transformation, conservation laws (including Noether theorems), Hamilton-Jacoby Equations, etc. - are preserved in this modified scheme and which parts require modifications to stay true. These and some other related questions are studied in the present work.

We will also show that any system of PDE of the form: "Euler- Lagrange equations with sources"

$$E_j(L) = f_j \tag{1.1}$$

can be realized by the Lagrangian formalism with a conventional action functional $\mathcal{A}(L)$ and the non-commuting variations defined by an appropriately chosen (defined by the sources f_j) tensor K of NC-variations. We show that the basic methods of conventional Lagrangian formalism - Noether Theorem, second variation technique, Hamiltonian equations, Weyl fields preserve their form in Lagrangian formalism with NC-variations. We study the relations between the properties of sources f_j and the curvature $\tilde{R}$ of the vertical connection tensor K.

We demonstrate that a variety of geometrical structures that appeared in the study of dynamics in some physical systems - dissipative potentials, non-holonomic transformations, torsion of zero curvature connections (absolute parallelism), material time and thermasy (= heat displacement), introduced by H.Helmholtz and studied by D.van Dantzig ([135, 136, 110]) are special cases or are closely related to the use of non-commutative variations defined by a vertical connection in the conventional Lagrangian formalism.

Our perspective in this work, supported by the results of the geometrical (bundle) form of Variational Calculus, is that the conventional rules of taking variations of the derivatives of dynamical variables (fields) (underlying the flow prolongation method) have important mathematical advantages (preservation of Cartan distribution, preservation of Lie bracket, etc.) making them more fundamental. Yet, a more general approach allows inclusion into the framework of Variational Calculus, the physical systems that can not be described by the conventional Lagrangian formalism.

In that we adopt the point of view of B. Vujanovich and that of A.Lurie's that in difference to the variations of fields y^i, variations of their derivatives $y^i_{,\mu}$ *are not only kinematical, but dynamical notions* and should be dealt with as such. In particular, this allows us to introduce geometrical factors that have dynamical meaning into the definition of variations of derivatives that have dynamical meaning. This allows us to describe dissipative processes in the system.

These notes are based on the Lectures delivered by author at the 15th Summer School in Global Analysis at Masaruk University,Brno, CZ on August 8-12, 2011. I am using this case to thank participants of this school and, especially, its organizer Professor Demeter Krupka and Dr.Marcella Palese for useful discussions during the school.

In particular, during this school Marcella Palese informed me about a non-conventional procedures of the non-commuting variations introduced by C.Murial and J.Romero in Spain and used by the group of specialists in Spain and Italy with the goal to extend the range of symmetry groups of Lagrangian systems. Their goals were different from ours but their constructions (of λ and μ-prolongations) are similar, but not identical, to our approach of using vertical connections. We have included a condensed exposition of their work and the relations with our scheme in the present text (see Chapter 4).

Preliminary results on the NC-variations Lagrangian formalism where published in the Proceedings of the GCM2008, [112].

2. Introduction.

In Chapter I we give a short sketch of classical Lagrangian formalism. Here we tried to make a presentation as simple as possible. Yet, we introduce in the beginning of Chapter 1, some basic invariant notions, whose more detailed description reader can find in the Appendix (and, in more details, in the literature refered there to. We define the configuration bundle, $\pi : Y \to X$, one-jet bundle $J^1(\pi)$ as the domain of Lagrangian functions and the total derivatives d_μ on the space $J^1(\pi)$ used to write down the Euler-Lagrange equations in an invariant way.

In Chapter 2 we define the non-commutative variations for an action functional of a Lagrangian $L(x^\mu, y^i, y^i_{,x^\mu})$ of the first order. Non-commutative variations are defined using tensors $K^\mu_{i\nu}$ in the configurational space Y. Variations of derivatives defined with the help of tensor K will be called **K-twisted variations** and the Euler-Lagrange equations obtained using variations defined this way will be called K-twisted EL-equations.

We get the Euler-Lagrange equations

$$EL(L)_\mu = f_\mu,\ \mu = 1, \dots, m$$

with the sources f_j defined by a "tensor K", formulate corresponding Noether Theorem (proved in Appendix III), present the canonical Energy-Momentum balance law.

A variety of examples of $EL + NV$ systems and classes of such systems are presented here.

Using Legendre transformation we construct corresponding Hamilton equations with sources and compare them with the "metriplectic or "double bracket" systems.

Then we show the form taken by the second variation formalism (sufficient conditions, Jacoby equation, etc.) in the case of NC-variations. At the end of this Chapter we show that this approach to the Lagrangian formalism can be readily extended to the higher order Lagrangian problems and to the case of "degenerate Lagrangians" where the source terms are of higher order then the Lagrangian itself.

In Chapter 3, we show that the procedure of K-twisted prolongation of a variation $\xi = \xi^i \partial_{y^\mu}$ of dynamical fields y^μ to the 1-jet bundle $J^1(\pi)$ is lacking two basic properties of the conventional flow prolongations of variational vector fields: conservation of Lie vector fields brackets and preservation of Cartan distribution in the 1-jet space. While the second property is valid only if tensor K vanishes, obstruction to the preservation of the Lie brackets is determined. It consists in two parts - curvature form tensor $\tilde{R}$ and the "skew-symmetric bracket" presenting the deformation of the Lie bracket of the vector fields.

Next, we show that the tensor K defining the Non-Commuting Lagrangian formalism has the form of the vertical component of an (Ehresmann) connection ω on the affine bundle $\pi_{10} : J^1(\pi) \to Y$ - the component responsible for the term of the form $a^i_\mu \partial_{y^i_\mu}$ of the K-vertical lift of a vector field $\xi = \xi^\mu \partial_{x^\mu} + \xi^i \partial_{y^i}$.

More then this, vertical/vertical component of the curvature $R(\omega)$ coincide with the tensor $\tilde{R}$ mentioned above. It is shown that one can define the covariant flow prolongation of vector fields from Y to $J^1(Y)$ so that the K-twisted prolongation of a vector field coincide with the modified by K flow prolongation.

In Sec.20, we consider the case where tensor K does not depend on the derivatives of dynamical fields $K^i_{\mu j} \in C^\infty(Y)$. We calculate different quantities characterizing K-twisted prolongations for this case and study the relation between the form of source terms, f_j, and the properties of the "curvature" $\tilde{R}$.

In Chapter 4 we present a short description of works of the group os spanish and italian mathematicians developed the Theory of twisted prolongations of vector fields to the jet bundles in many respects similar to our scheme. Their works had different goal - to construct, using the twisted prolongations of vector fields, alternative classes of symmetry groups of differential equations and systems of differential equations. their "theory of λ and μ -prolongations and symmetries has an important property - vector fields obtained by the prolongations to the jet bundles preserves, in some modified sense, the Cartan distributions and contact formes. This property has an elegant form and probably can be useful in further development of Geometrical Theory of Differential Equations.

In Chapter 5, we discuss several situations when the non-commuting variations were used explicitly or implicitly in the variational description of some physical systems. For some time a "geometrization" of a mechanical system, i.e., presentation equations of motion of such a system as the geodesic motion with respect to some linear connection in the configurational space Q of this system was a very popular problem in Mechanics. In Sec.40, it is shown, following the work of B.Vujanovich,[134, 133] that the same result can be achieved without changing the geometry of the space Q but, instead, by using conventional Lagrangian of this system and redefining the variations of velocities in the tangent space $T(Q)$. In Sec.41, the short review of variational approach to the non-holonomic mechanical systems is presented. Using the approach of L.Boltzman we construct the equations of motion in non-holonomic systems with line non-holonomic relations. We notice that the Bolzmann tensor defining the non-commutativity tensor K of the variations is defined here by the torsion of the zero curvature connection corresponding to the non-holonomic frame (see Appendix I, Sec.66).

In Sections 42,43 we present the use of non-holonomic (gauge) transformations for constructing Variational principle with non-commuting variations defined by the torsion of the (absolute parallelism) connection given by this transformation. First example of such scheme (see Section 42) is the one that was developed by H.Kleinert and his collaborators P.Fiziev and A.Pelster [35, 65] to describe Mechanics in spaces with metrics and connections (Cartan spaces). In Section 43 we present a short review of properties of **Uniform Materials**. Uniform materials were defined by K.Kondo (1955) and developed by a variety of specialists including E.Kroner, B.Bilby, C.C.Wang, C.Truesdell in 60th of XX century and by many specialists later on. We refer to the monographs [22, 137] for the detailed description of "Uniform materials"" theory.

In Sec.44, we discuss the relation between the method of non-commuting variations and the use of **dissipative potentials** (special case of which are Rayleigh dissipative function) in Lagrangian formalism.

In Chapter 6, we present an application of Lagrangian formalism with the NC-variations to the description of irreversible evolution of a continuous media with heating and structural changes.

In Sec. (46) we introduce thermasy, a scalar variable introduced by H. Helmholtz and later on, used by D.van Dantzig in his study of thermodynamics of moving matter, see [135], A.Green and P.Nahdi in thermoelasticity, see [53, 54] and G.Maugin and V.Kalpakides in Continuum Thermodynamics, [92]. Using thermasy, whose time derivative is absolute temperature one can formulate **entropy balance of a thermodynamical system as the Euler-Lagrange Equation**. We present modified and simplified version of this variational system and write down corresponding energy balance and the heat propagation equation that has the form of Cattaneo heat propagation law, see [100].

Then we introduce the model of material metric space-time (P, G) that was introduced by A.Chudnovsky and the author in order to model the aging processes in the materials, [15, 16]. In this model, evolution of the material is presented by smooth embedding of the material space-time into the Galilean space-time and the material metric G describes the structural properties of material. In particular, the rate S of the proper (=material) time τ relative to the physical type: $d\tau = Sdt$ is the characteristic of the entropy production in the material (if entropy production is zero, $S = 1$). We show that the entropy balance in a thermodynamical system obtained as the Euler-Lagrange for thermasy using the NC-variations defined by the rate of material time S coincide with the Euler-Lagrange Equation for thermasy **obtained using the material time τ instead of physical time** t and **conventional variations instead of NC-variations**.

This duality shows that the usage of NC-variations allows us to model complex irreversible phenomena that is impossible to do using conventional Lagrangian approach.

Appendix:

In the Appendix I we present a short review of geometrical notions used in the text: manifolds, fiber bundles, connections and their curvature, linear connection and its rorsion, prolongation of vector fields from Y to the jet bundles, absolute parallelism. In appendix II we define jet bundles, their mappings, total derivatives, contact structure of jet bundles, connections in jet bundles, Lie vector fields, properties of vertical connections. In Appendix III we define the symmetries and infinitesimal symmetries of the differential systems and the Lagrangian action, define the Noether formal;ism and probe the the First Noether Theorem. In the case od Euler-lagrange equation with sources, Noether equations corresponding to the symmetry Lie groups are **balance equations** rather then the conservation laws. This referees, in particular, to the energy-momentum balance laws.