Part IV
Fixed Points for Families of Maps

In these final five chapters we'll turn our attention to fixed-point theorems involving, not just a single map, but a *family* of them for which we aim to produce a *common* fixed point. Necessarily we'll have to place severe restrictions on our classes of maps, but even so the results obtained will have surprising consequences that connect topology, algebra, and measure theory.

The fixed-point theorems we'll prove guarantee for every compact topological group the existence of Haar measure: a Borel probability measure that is invariant under the action of the group on itself. The model for this is arc-length measure on the unit circle, the group-invariance of which is the basis for Fourier analysis. The invariant measures we'll produce in the next few chapters play a similar role for the harmonic analysis of functions on compact groups, and we'll say something about how this goes in the abelian case.

An equally important thread involves the use of fixed-point theorems to produce *finitely additive* "measures" that are invariant under certain groups of transformations. This will lead us into the study of "paradoxical decompositions," the most famous example being the Banach–Tarski Paradox, which asserts that the unit ball of $\mathbb{R}^3$ can be split up into a finite number subsets that can be reassembled, using only rigid motions, into *two* unit balls. We'll spend some time understanding this paradox, and will show, via fixed-point theorems, that nothing similar is possible for either the unit circle or the unit disc.