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A Fixed-Point Farrago

 Springer

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For Lilah

Farrago—noun

An assortment or a medley; a conglomeration.

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Preface

Fixed points show up everywhere in mathematics. This book provides an introduction to some of the subject's best-known theorems and some of their most important applications, emphasizing throughout their interaction with topics in the analysis familiar to students of mathematics. The level of exposition increases slowly, requiring at first some undergraduate-level proficiency, then gradually increasing to the kind of sophistication one might expect from a graduate student. Appendices at the back of the book provide introduction to (or reminder of) some of the prerequisite material. To encourage active participation, exercises are integrated into the text. Thus I hope readers will find the book reasonably self-contained and useable either on its own or as a supplement to standard courses in the mathematics curriculum.

The material is split into four parts, the first of which introduces the Banach Contraction Mapping Principle and the Brouwer Fixed-Point theorem, along with a selection of interesting applications: Newton's method, initial value problems, and positive matrices (e.g., the Google matrix). Brouwer's theorem is proved in dimension two via Sperner's lemma, and Banach's principle is proved in generality. Included also is a lesser known fixed-point theorem due to Knaster and Tarski—an easy-to-prove result about functions taking sets to sets that makes short work of the Schröder–Bernstein theorem and plays an important role in a later chapter on paradoxical decompositions.

Part II focuses on Brouwer's theorem, featuring an analysis-based proof of the general result, and John Nash's application of this result to the existence of Nash equilibrium. Brouwer's theorem leads to Kakutani's theorem on set-valued maps, upon which rests Nash's remarkable "one-page" proof of his famous theorem. A brief introduction to game theory motivates the exposition of Nash's results.

The material of these first two parts should be accessible to undergraduates whose background includes the standard junior–senior-level courses in linear algebra and analysis taught at American colleges, which hopefully provides some familiarity with basic set theory and metric spaces.

Part III applies Brouwer's theorem to spaces of infinite dimension, where it provides an important step in the proof of the Schauder Fixed-Point theorem. Schauder's theorem leads to both Peano's existence theorem for initial value prob-

lems and Lomonosov's spectacular theorem on invariant subspaces for linear operators on Banach spaces. For this segment the reader needs only some experience with the basics of Hilbert and Banach spaces.

The fourth and final part of the book rests on the work of Markov, Kakutani, and Ryll-Nardzewski concerning fixed points for families of affine maps. These results lead to the existence of measures—both finitely and countably additive—that are invariant under various groups of transformations. In the finitely additive case, this leads to the concepts of invariant means and “paradoxical decompositions,” especially the Banach–Tarski paradox. The countably additive case leads to the existence of Haar measure on compact topological groups. This part of the book gets into notions of duality and weak-star topologies, with the necessary prerequisites developed from scratch—but only within the narrow context in which they are used. The result is a gentle introduction to abstract duality which suffices for our purposes, and hopefully encourages the reader to appreciate this way of thinking.

Much of the material presented here originated in lectures given during the academic years 2012–2013 by participants in the Analysis Seminar at Portland State University. I am particularly indebted to John Erdman, who organized the seminar for many years and who introduced us to the Knaster–Tarski theorem; to Steve Silverman who lectured on the work of Markov and Kakutani; to Mau Nam Nguyen, Blake Rector, and Jim Rulla for their talks on set-valued analysis; to Steve Bleiler and Cody Fuller for their lectures on game theory; and to all the seminar participants whose thoughtful questions and comments contributed greatly to my appreciation of the subject.

Sheldon Axler encouraged me to turn my lecture notes into a book and suggested that Sperner's lemma might have a place in it. Paul Bourdon contributed many insightful comments on initial versions of manuscript and cleaned up several of my more cumbersome arguments. The Fariborz Maseeh Department of Mathematics and Statistics at Portland State University provided office space and technical assistance, and Michigan State University—my employer in a former life—provided invaluable electronic access to its library.

Above all, I owe a deep debt of gratitude to my wife Jane, without whose understanding and encouragement this project could never have reached completion.

Portland, OR, USA
August 2015

Joel H. Shapiro

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