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Tempered Stable Distributions

Stochastic Models for Multiscale Processes

Springer

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To Lijuan

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Notation

Let $\mathbb{R}^d$ be the space of d -dimensional column vectors of real numbers. For $x \in \mathbb{R}^d$ we write $x = (x_1, x_2, \dots, x_d)$, and we denote the transpose of x by x^T . Let $\langle \cdot, \cdot \rangle$ be the usual inner product on $\mathbb{R}^d$, i.e., if $x, y \in \mathbb{R}^d$ then $\langle x, y \rangle = x^T y = \sum_{i=1}^d x_i y_i$. Let $|\cdot|$ be the usual norm on $\mathbb{R}^d$, i.e., if $x \in \mathbb{R}^d$ then $|x| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^d x_i^2}$. Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$, and let $\mathbb{C}^d$ be the space of d -dimensional column vectors of complex numbers. For $z \in \mathbb{C}^d$ we write $\Re z$ to denote the real part of z and $\Im z$ to denote the imaginary part of z . Let $\mathbb{R}^{d \times d}$ be the collection of all $d \times d$ -dimensional matrices with real entries, and for $A \in \mathbb{R}^{d \times d}$ let $\text{tr} A$ be the trace of A . Let $\mathfrak{B}(\mathbb{R}^d)$ and $\mathfrak{B}(\mathbb{S}^{d-1})$ denote the Borel sets on $\mathbb{R}^d$ and $\mathbb{S}^{d-1}$, respectively.

If μ is a probability measure on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$, we write $\hat{\mu}(z) = \int_{\mathbb{R}^d} e^{i\langle x, z \rangle} \mu(dx)$ for $z \in \mathbb{R}^d$ to denote its characteristic function, and we write $X \sim \mu$ to denote that X is a random variable taking values in $\mathbb{R}^d$ with distribution μ . Further, we write $X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \mu$ to denote that $X_1, X_2, \dots, X_n$ are independent and identically distributed random variables taking values in $\mathbb{R}^d$, each with distribution μ . For two random variables X and Y , we write $X \stackrel{d}{=} Y$ to denote that X and Y have the same distribution.

For a sequence of random variables $X_1, X_2, \dots$, we write $\text{d-lim } X_n$ to denote the limit in distribution. We also use the more standard notations $\xrightarrow{d}$ and $\xrightarrow{p}$ to denote, respectively, convergence in distribution and convergence in probability. For sequences of probability measures, we write $\xrightarrow{w}$ to denote weak convergence, and for sequences of Radon measures, we write $\xrightarrow{v}$ to denote vague convergence.

We write $ID(A, M, b)$ to denote the infinitely divisible distribution on $\mathbb{R}^d$ with Gaussian part A , Lévy measure M , and shift b . We write $N(b, A)$ to denote the Gaussian distribution on $\mathbb{R}^d$ with mean vector b and covariance matrix A , and we write $S_\alpha(\sigma, b)$ to denote the infinite variance α -stable distribution on $\mathbb{R}^d$ with spectral measure σ and shift b . We write $TS_\alpha^p(R, b)$ to denote the p -tempered α -stable distribution on $\mathbb{R}^d$ with Rosiński measure R and shift b , and we write $ETS_\alpha^p(A, \nu, b)$ to denote the extended p -tempered α -stable distribution on $\mathbb{R}^d$ with Gaussian part A , extended Rosiński measure ν , and shift b . Several parametric families of

tempered stable distributions are introduced in Chapter 6. These are denoted by $STLF_\alpha^p(c_-, c_+, \ell_-, \ell_+, b)$, $TW_\alpha^p(c, \ell, b)$, $TW_\alpha(c, \ell, b)$, $PT_\alpha^p(c_-, c_+, \nu_-, \nu_+, b)$, and $GT_\alpha^p(c_-, c_+, \nu_-, \nu_+, \ell_-, \ell_+, b)$.

If $\mathcal{A}$ is a collection of subsets of some space $\mathbb{E}$, we write $\sigma(\mathcal{A})$ to denote the σ -algebra generated by $\mathcal{A}$, i.e., the smallest σ -algebra that contains $\mathcal{A}$. If f and g are real-valued functions, $c \in \{0, \infty\}$, and $k \in \mathbb{R}$, we write

$$f(t) \sim kg(t) \text{ as } t \rightarrow c$$

to denote

$$f(t)/g(t) \rightarrow k \text{ as } t \rightarrow c.$$

Note that this is somewhat nonstandard notation in the case when $k = 0$. When dealing with infinity we adopt the conventions that

$$1/\infty = 0 \text{ and } 1/0 = \infty.$$

For $x, y \in \mathbb{R}$ we write $x \wedge y$ to denote the minimum and $x \vee y$ to denote the maximum. Further, we define $\log^+ x := \log(1 \vee x)$. For a set H we denote the indicator function on H by 1_H . This means that $1_H(x) = 1$ if $x \in H$ and $1_H(x) = 0$ if $x \notin H$. We also write $1_{x \in H}$ to denote this. For $a \in \mathbb{R}^d$ we write δ_a to denote the Dirac delta measure at a . This means that for any Borel set H , we have $\delta_a(H) = 1_H(a)$. We denote the gamma function by $\Gamma(x)$. When $x > 0$ we have

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

Further, we extend the gamma function to any $x \in \mathbb{R}$ with $x \neq 0, -1, -2, \dots$ by the relation

$$\Gamma(x+1) = x\Gamma(x).$$

If P is a probability measure on the measurable space $(\Omega, \mathcal{F})$, we write E_P to denote the expectation with respect to P . Further, when there is no chance of ambiguity, we write E instead of E_P .