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Integral Operators in Non-Standard Function Spaces

Volume 1: Variable Exponent Lebesgue
and Amalgam Spaces

 Birkhäuser

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Preface

This book is a result of our ten-year fruitful collaboration. It deals with integral operators of harmonic analysis and their various applications in new, non-standard function spaces. Specifically, we deal with variable exponent Lebesgue and amalgam spaces, variable exponent Hölder spaces, variable exponent Campanato, Morrey and Herz spaces, Iwaniec–Sbordone (grand Lebesgue) spaces, grand variable exponent Lebesgue spaces, which unify two types of spaces mentioned above, grand Morrey spaces, generalized grand Morrey spaces, as well as weighted analogues of most of them.

In recent years it was realized that the classical function spaces are no longer appropriate spaces when we attempt to solve a number of contemporary problems arising naturally in: non-linear elasticity theory, fluid mechanics, mathematical modelling of various physical phenomena, solvability problems of non-linear partial differential equations. It thus became necessary to introduce and study the spaces mentioned above from various viewpoints. One of such spaces is the variable exponent Lebesgue space. For the first time this space appeared in the literature already in the thirties of the last century, being introduced by W. Orlicz. In the beginning these spaces had theoretical interest. Later, at the end of the last century, their first use beyond the function spaces theory itself, was in variational problems and studies of $p(x)$ -Laplacian, in Zhikov [375, 377, 376, 379, 378], which in its turn gave an essential impulse for the development of this theory. The extensive investigation of these spaces was also widely stimulated by appeared applications to various problems of Applied Mathematics, e.g., in modelling electrorheological fluids Acerbi and Mingione [3], Rajagopal and Růžička [301], Růžička [306] and more recently, in image restoration Aboulaich, Meskine, and Souissi [1], Chen, Levine, and Rao [42], Harjulehto, Hästö, Latvala, and Toivanen [127], Rajagopal and Růžička [301].

Variable Lebesgue space appeared as a special case of the Musielak–Orlicz spaces introduced by H. Nakano and developed by J. Musielak and W. Orlicz.

The large number of various results for non-standard spaces obtained during last decade naturally led us to two-volume edition of our book. In this Preface to Volume 1 we briefly characterize the book as a whole, and provide more details on the material of Volume 1.

Recently two excellent books were published on variable exponent Lebesgue spaces, namely:

L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, Vol. 2017, Springer, Heidelberg, 2011,

and

D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces, Foundations and Harmonic Analysis*, Birkhäuser, Springer, Basel, 2013.

A considerable part of the first book is devoted to applications to partial differential equations (PDEs) and fluid dynamics. In the recent book

V. Kokilashvili and V. Paatashvili, *Boundary Value Problems for Analytic and Harmonic Functions in Non-standard Banach Function Spaces*, Nova Science Publishers, New York, 2012,

there are presented applications to other fields, namely to boundary value problems, including the Dirichlet, Riemann, Riemann–Hilbert and Riemann–Hilbert–Poincaré problems. These problems are solved in domains with non-smooth boundaries in the framework of weighted variable exponent Lebesgue spaces.

The basic arising question is: what is the difference between this book and the above-mentioned books? What new theories and/or aspects are presented here? What is the motivation for a certain part of the book to treat variable exponent Lebesgue spaces? Below we try to answer these questions.

First of all, we claim that most of the results presented in our book deal with the integral transforms defined on general structures, namely, on measure metric (quasi-metric) spaces. A characteristic feature of the book is that most of statements proved here have the form of criteria (necessary and sufficient conditions).

In the part related to the variable exponent Lebesgue spaces in Volume 1 we single out the results for: weighted inequality criteria for Hardy-type and Carleman–Knopp operators, a weight characterization of trace inequalities for Riemann–Liouville transforms of variable order, two-weight estimates, and a solution of the trace problem for strong fractional maximal functions of variable order and double Hardy transforms. It should be pointed out that in this problem the situation is completely different when the fractional order is constant. Here two-weight estimates are derived without imposing the logarithmic condition for the exponents of spaces. We also treat boundedness/compactness criteria for weighted kernel operators including, for example, weighted variable-order fractional integrals.

For the variable exponent amalgam spaces we give a complete description of those weights for which the corresponding weighted kernel operators are bounded/compact. The latter result is new even for constant exponent amalgam spaces. We

give also weighted criteria for the boundedness of maximal and potential operators in variable exponent amalgam spaces.

In Volume 1 we also present the results on mapping properties of one-sided maximal functions, singular, and fractional integrals in variable exponent Lebesgue spaces. This extension to the variable exponent setting is not only natural, but also has the advantage that it shows that one-sided operators may be bounded under weaker conditions on the exponent those known for two-sided operators. Among others, two-weight criterion is obtained for the trace inequality for one-sided potentials.

In this volume we state and prove results concerning mapping properties of hypersingular integral operators of order less than one in Sobolev variable exponent spaces defined on quasi-metric measure spaces. High-order hypersingular integrals are explored as well and applied to the complete characterization of the range of Riesz potentials defined on variable exponent Lebesgue spaces.

Special attention is paid to the variable exponent Hölder spaces, not treated in existing books. In the general setting of quasi-metric measure spaces we present results on mapping properties of fractional integrals whose variable order may vanish on a set of measure zero. In the Euclidean case our results hold for domains with no restriction on the geometry of their boundary.

The established boundedness criterion for the Cauchy singular integral operator in weighted variable exponent Lebesgue spaces is essentially applied to the study of Fredholm type solvability of singular integral equations and to the PDO theory. Here a description of the Fredholm theory for singular integral equations on composite Carleson curves oscillating near nodes, is given using Mellin PDO.

In Volume 2 the mapping properties of basic integral operators of Harmonic Analysis are studied in generalized variable exponent Morrey spaces, weighted grand Lebesgue spaces, and generalized grand Morrey spaces. The grand Lebesgue spaces are introduced on sets of infinite measure and in these spaces boundedness theorems for sublinear operators are established. We introduce new function spaces unifying the variable exponent Lebesgue spaces and grand Lebesgue spaces. Boundedness theorems for maximal functions, singular integrals, and potentials in grand variable exponent Lebesgue spaces defined on spaces of homogeneous type are established.

In Volume 2 the grand Bochner–Lebesgue spaces are introduced and some of their properties are treated.

The entire book is mostly written in the consecutive way of presentation of the material, but in some chapters, for reader's convenience, we recall definitions of some basic notions. Although we use a unified notation in most of the cases, in some of the cases the notation in a chapter is specific for that concrete chapter.

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Contents

Volume 1

Preface	v
1 Hardy-type Operators in Variable Exponent Lebesgue Spaces	
1.1 Preliminaries	2
1.1.1 Definitions and Basic Properties	2
1.1.2 Equivalent Norms	4
1.1.3 Minkowski Integral Inequality	5
1.1.4 Basic Notation	5
1.1.5 Estimates for Norms of Characteristic Functions of Balls	6
1.2 Convolution Operators	7
1.2.1 Convolution Operators Bounded in $L^{p(\cdot)}(\mathbb{R}^n)$	7
1.2.2 Estimation of Norms of Some Embeddings for Variable Exponent Lebesgue Spaces	9
1.2.3 Estimation of the Norm of Convolution Operators	13
1.3 Reduction of Hardy Inequalities to Convolution Inequalities	15
1.3.1 Equivalence Between Mellin Convolution on \mathbb{R}_+ and Convolutions on \mathbb{R} . The Case of Constant p	15
1.3.2 The Case of Variable p	16
1.4 Variable Exponent Hardy Inequalities	17
1.5 Estimation of Constants in the Hardy Inequalities	20
1.6 Mellin Convolutions in Variable Exponents Spaces $L^{p(\cdot)}(\mathbb{R}_+)$	23
1.7 Knopp–Carleman Inequalities in the Variable Exponent Setting	24
1.8 Comments to Chapter 1	26

2 Oscillating Weights

2.1	Preliminaries	28
2.2	Oscillating Weights of Bari–Stechkin Class	30
2.2.1	Some Classes of Almost Monotone Functions	30
2.2.2	ZBS Classes and MO Indices of Weights at the Origin	31
2.2.3	Examples of Weights	36
2.2.4	ZBS Classes and MO Indices of Weights at Infinity	36
2.3	Maximal Operator with Oscillating Weights	38
2.3.1	Weighted Pointwise Estimates	39
2.3.2	Weighted Boundedness; the Euclidean Case	42
2.3.3	A Non-Euclidean Case	48
2.4	Weighted Singular Operators	50
2.4.1	Calderón–Zygmund-type Operators: the Euclidean Case	51
2.4.2	Singular Integrals with General Weights on Lyapunov Curves	53
2.4.3	Preliminaries Related to Singular Integrals on Curves	55
2.4.4	Singular Integrals with Cauchy Kernel on Carleson Curves	59
2.4.5	The Property of Γ to be a Carleson Curve is Necessary	63
2.5	Weighted Potential Operators	67
2.5.1	Non-weighted Estimates in the Prelimiting Case; the Euclidean Version	68
2.5.2	Non-weighted Estimates in the Limiting Case; the Euclidean Version	69
2.5.3	Non-weighted Estimates in the Prelimiting Case on Quasimetric Measure Spaces	73
2.5.4	Weighted Norm Estimates of Truncated Potential Kernels in the Euclidean Case	80
2.5.5	Fractional Integrals on Bounded Sets $\Omega \subset \mathbb{R}^n$ with Oscillating Weights and Variable Order $\alpha(x)$	84
2.5.6	Fractional Integrals on \mathbb{R}^n with Power Weights Fixed at the Origin and Infinity and Constant α	89
2.5.7	Spherical Fractional Integrals on \mathbb{S}^n with Power Weights	93
2.6	Generalized Potentials	95
2.6.1	Preliminaries	98
2.6.2	Estimation of the Variable Exponent Norm of Truncated Generalized Potentials	100
2.6.3	An Appropriate Φ -Function	103
2.6.4	Proof of Theorem 2.72	104

2.6.5	Weighted Version	105
2.7	Weighted Extrapolation	106
2.7.1	Preliminaries Related to Quasimetric Measure Spaces . . .	106
2.7.2	Classes of the Weight Functions	109
2.7.3	Extrapolation Theorem	111
2.8	Application to Boundedness Problems in $L^{p(\cdot)}(\Omega, \varrho)$ for Classical Operators of Harmonic Analysis	115
2.8.1	Potential Operators and Fractional Maximal Function . .	115
2.8.2	Fourier Multipliers	116
2.8.3	Multipliers of Trigonometric Fourier Series	119
2.8.4	Majorants of Partial Sums of Fourier Series	121
2.8.5	Cauchy Singular Integral	121
2.8.6	Multidimensional Singular-type Operators	122
2.8.7	Fefferman–Stein Function and Vector-valued Operators . .	123
2.9	Comments to Chapter 2	124
3	Kernel Integral Operators	
3.1	Preliminaries	130
3.1.1	Variable Exponent Lebesgue Spaces	130
3.1.2	Variable Exponent Amalgam Spaces (VEAS)	135
3.1.3	Two-weighted Hardy Operator on the Real Line	137
3.1.4	Some Discrete Inequalities	141
3.2	Kernel Operators in $L^{p(\cdot)}$ Spaces	142
3.2.1	Boundedness Criteria	142
3.2.2	Compactness	146
3.2.3	Measure of Non-compactness	150
3.2.4	The Riemann–Liouville Operator with Variable Parameter	154
3.3	Boundedness in Variable Exponent Amalgam Spaces	156
3.3.1	General Operators on Amalgams	156
3.3.2	Two-weighted Hardy Operator	164
3.3.3	Kernel Operators in $(L^{p(\cdot)}(\mathbb{R}_+), l^q)_d$ and $(L^{p(\cdot)}(\mathbb{R}), l^q)$. . .	166
3.4	Maximal Functions and Potentials on VEAS	170
3.4.1	Maximal Operators in $(L^{p(\cdot)}(\mathbb{R}), l^q)$	184
3.4.2	Fractional Integrals. Trace Inequality	188
3.5	Compactness of Kernel Operators on VEAS	190
3.6	Product Kernel Integral Operators with Measures	197
3.6.1	Hardy Operator with Respect to a Measure	198
3.6.2	Main Results	200
3.6.3	Proofs of the Main Results	205
3.6.4	A Fefferman–Stein-type Inequality	213

3.7	Comments to Chapter 3	215
4	Two-weight Estimates	
4.1	Preliminaries	219
4.1.1	Some Properties of Variable Exponent Lebesgue Spaces	219
4.1.2	Variable Exponent Lebesgue Space on Quasimetric Measure Spaces	220
4.1.3	Carleson–Hörmander Inequality	237
4.2	A Sawyer-type Condition on a Bounded Interval	239
4.3	A Sawyer-type Condition on an Unbounded Interval	246
4.4	Hardy-type Operators on Quasimetric Measure Spaces	249
4.5	Modular Conditions for Fractional Integrals	260
4.6	Modular Conditions for Maximal and Singular Operators	270
4.7	Norm-type Conditions for Maximal and Calderón–Zygmund Operators	278
4.7.1	Maximal Functions and Singular Integrals on SHT	278
4.7.2	Maximal Functions and Singular Integrals on \mathbb{R}_+	284
4.8	Potentials with Variable Parameters	286
4.8.1	Weighted Criteria for Potentials	288
4.8.2	Applications to Gradient Estimates	292
4.8.3	Potentials on Fractal Sets	294
4.9	Comments to Chapter 4	295
5	One-sided Operators	
5.1	Preliminaries	298
5.2	One-sided Extrapolation	302
5.3	One-sided Maximal Functions	304
5.4	One-sided Potentials	317
5.5	One-sided Calderón–Zygmund Operators	321
5.6	Weighted Criteria for One-sided Operators	323
5.6.1	Hardy–Littlewood One-sided Maximal Functions. One-weight Inequality	323
5.6.2	One-sided Fractional Maximal Operators. One-weight Inequality	327
5.7	Generalized One-sided Fractional Maximal Operators	331
5.7.1	The Two-weight Problem	331
5.7.2	Fefferman–Stein-type Inequalities	342
5.8	Two-weight Inequalities for Monotonic Weights	343
5.9	The Riemann–Liouville Operator on the Cone of Decreasing Functions	351
5.10	Comments to Chapter 5	354

6 Two-weight Inequalities for Fractional Maximal Functions

6.1 Preliminaries 356

6.2 Generalized Maximal Function and Potentials 358

6.2.1 Fractional Maximal Function 358

6.2.2 Fractional Integrals 364

6.2.3 Diagonal Case 366

6.2.4 Further Remarks 368

6.3 Fractional Integral Operators on the Upper Half-space 370

6.3.1 Non-diagonal Case 371

6.3.2 Diagonal Case 378

6.4 Double Hardy Operator 381

6.5 Strong Fractional Maximal Functions in $L^{p(\cdot)}$ Spaces.
Unweighted Case 386

6.6 Two-weight Estimates for Strong Fractional Maximal
Functions 388

6.6.1 Formulation of Results 389

6.6.2 Proofs 390

6.7 Comments to Chapter 6 394

7 Description of the Range of Potentials

7.1 Preliminaries on Higher-order Hypersingular Integrals 395

7.2 Denseness of the Lizorkin Test Functions Space in $L^{p(\cdot)}(\mathbb{R}^n)$. . . 396

7.3 Inversion of the Riesz Potential Operator on the
Space $L^{p(\cdot)}(\mathbb{R}^n)$ 398

7.4 Characterization of the Space of Riesz and Bessel Potentials
of Functions in $L^{p(\cdot)}(\mathbb{R}^n)$ 400

7.4.1 Preliminaries 400

7.4.2 Characterization of the Riesz Potentials on
 $L^{p(\cdot)}$ -Spaces 401

7.5 Function Spaces Defined by Fractional Derivatives
in $L^{p(\cdot)}(\mathbb{R}^n)$ 403

7.5.1 Definitions 403

7.5.2 Denseness of C_0^∞ in $L^{p(\cdot),\alpha}(\mathbb{R}^n)$ 403

7.6 Bessel Potentials Space of Functions in $L^{p(\cdot)}(\mathbb{R}^n)$ and
its Characterization 410

7.6.1 Basic Properties 410

7.6.2 Characterization of the Space $B^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ via
Hypersingular Integrals 410

7.6.3 Proof of Lemmas 7.14 and 7.15 413

7.7 Connection of the Riesz and Bessel Potentials with the
Sobolev Variable Exponent Spaces 416

7.7.1	Coincidence with Variable Exponent Sobolev Spaces for $\alpha \in \mathbb{N}$	416
7.7.2	Denseness of C_0^∞ -Functions in $W^{1,p(\cdot)}(\mathbb{R}^n)$	418
7.8	Characterization of the Variable Exponent Bessel Potential Space via the Rate of Convergence of the Poisson Semigroup	423
7.8.1	More on Fourier $p(x)$ -Multipliers	424
7.8.2	On Finite Differences	425
7.8.3	More on the Function $K_{\ell,\alpha}(x)$	426
7.8.4	Crucial Lemmas	427
7.8.5	$A(x)$ and $B(x)$ are Fourier $p(\cdot)$ -Multipliers	431
7.8.6	Main Theorems	435
7.9	Comments to Chapter 7	437
8	Embeddings into Hölder Spaces	
8.1	Preliminaries on Hypersingular Integrals	439
8.2	Embeddings of Variable Sobolev Spaces into Hölder Spaces: the Euclidean Case	441
8.2.1	Hölder Spaces of Variable Order	441
8.2.2	Pointwise Inequalities for Sobolev Functions	442
8.2.3	Embedding Theorems for Hajlasz–Sobolev spaces	443
8.2.4	Extension to Higher Smoothness	444
8.3	Embeddings into Hölder Function Spaces on Quasimetric Measure Spaces	446
8.3.1	Variable Exponent Hölder Spaces on Quasimetric Spaces	446
8.3.2	Variable Exponent Hajlasz–Sobolev Spaces	447
8.3.3	Embeddings of Variable Exponent Hajlasz–Sobolev Spaces	448
8.3.4	Hypersingular Integrals in Variable Exponent Hajlasz–Sobolev Spaces	450
8.4	Comments to Chapter 8	453
9	More on Compactness	
9.1	Two General Results on Compactness of Operators	455
9.1.1	Dominated Compactness Theorem	455
9.1.2	Compactness under Interpolation Theorem	460
9.1.3	Compactness of an Integral Operator with Integrable Almost Decreasing Radial Dominant Majorant of the Kernel in the Case $ \Omega < \infty$	461
9.2	The Case $\Omega = \mathbb{R}^n$: Compactness of Convolution-type Operators with Coefficients Vanishing at Infinity	464
9.3	Comments to Chapter 9	465

10 Applications to Singular Integral Equations

10.1 Singular Integral Equations with Piecewise Continuous Coefficients 468

10.1.1 Introduction 468

10.1.2 Statement of the Main Result for the Spaces $L^{p(\cdot)}(\Gamma)$. . . 469

10.1.3 Singular Integral Operators in Banach Function Spaces $X(\Gamma)$ 470

10.1.4 Proof of Theorem 10.4 478

10.2 Pseudodifferential Operators 478

10.2.1 Boundedness in $L^{p(\cdot)}(\mathbb{R}^n, w)$ of Singular Integral-type Operators 479

10.2.2 On Calculus of PDO on \mathbb{R}^n 484

10.2.3 Operators with Slowly Oscillating Symbols 485

10.2.4 Boundedness of PDO in $H^{s,p(\cdot)}(\mathbb{R}^n)$ 486

10.2.5 Fredholmness of PDO in $L^{p(\cdot)}(\mathbb{R}^n)$ and $H^{s,p(\cdot)}(\mathbb{R}^n)$ 487

10.2.6 Pseudodifferential Operators with Analytic Symbols in the Space $H^{s,p(\cdot)}(\mathbb{R}^n)$ 494

10.3 Singular Integral Equations on Composite Carleson Curves via Mellin PDO 497

10.3.1 Introduction 497

10.3.2 Pseudodifferential Operators on \mathbb{R} 499

10.3.3 Mellin Pseudodifferential Operators 508

10.3.4 Singular Integral Operators on Some Classes of Carleson Curves 513

10.3.5 Comparison of the Used Class of Oscillating Weights with the Bari–Stechkin-type Weights 525

10.4 Comments to Chapter 10 527

Bibliography 529

Symbol Index 555

Author Index 559

Subject Index 565

Volume 2

Preface	v
Basic Definitions and Notation from Volume 1	xix

Part I Hölder Spaces of Variable Order

11 Variable exponent Hölder Spaces

11.1 Preliminaries	572
11.1.1 Notation	572
11.1.2 Two Technical Lemmas	573
11.1.3 Estimation of Truncated Potential-type Integrals Via One-dimensional Integrals	576
11.1.4 Hölder and Generalized Hölder Spaces with Variable Characteristics on a Quasimetric Measure Space	580
11.1.5 Zygmund–Bari–Stechkin Classes $\Phi_{\beta(\cdot)}^{\delta(\cdot)}$ Depending on a Parameter x	582
11.2 Potentials and Hypersingular Integrals	584
11.2.1 Zygmund-type Estimates of Potentials	584
11.2.2 Zygmund-type Estimates for Hypersingular Integrals	589
11.2.3 Mapping Properties of Potentials and Hypersingular Operators of Variable Order in the Spaces $H^{w(\cdot)}(\Omega)$	591
11.3 Potentials of Constant Order on Sets without Cancellation Property in Variable Hölder Spaces	593
11.3.1 Potentials of Constant Functions	594
11.3.2 On the α -Property of Sets	596
11.3.3 Mapping Properties of the Potential Operator I^α in Generalized Hölder Spaces	599
11.3.4 The Case of Spatial and Spherical Potentials in \mathbb{R}^n	600
11.4 Comments to Chapter 11	603

Part II Variable Exponent Morrey–Campanato and Herz Spaces

12 Morrey-type Spaces; Constant Exponents

12.1 Interrelations Between Morrey and Stummel Spaces	607
12.1.1 Notation and Definitions	607
12.1.2 Weighted Integrability of Functions in Generalized Local Morrey Spaces	609
12.1.3 Stummel Spaces	612

12.1.4	Embeddings for Global Morrey Spaces	616
12.2	Stein–Weiss-type Theorems in L^p Spaces	617
12.3	Potentials Defined by Measures: Classical Morrey Spaces	629
12.3.1	Preliminaries	630
12.3.2	Hardy-type Inequalities	631
12.3.3	Fractional Integrals Defined on Spaces of Homogeneous Type	633
12.4	Comments to Chapter 12	640
13	Morrey, Campanato and Herz Spaces with Variable Exponents	
13.1	Hardy-type Operators in Variable Exponent Morrey Spaces	644
13.1.1	Introduction	644
13.1.2	Preliminaries on Lebesgue $p(\cdot)$ -Norms	644
13.1.3	Variable Exponent Morrey Spaces	647
13.1.4	Some Weighted Estimates of Functions in Morrey Spaces	653
13.1.5	Weighted Hardy Operators in Generalized Morrey Spaces	655
13.1.6	Finding $\psi(0, r)$ by a Given $\varphi(0, r)$	658
13.2	Hardy-type Operators in Vanishing Morrey Spaces	661
13.2.1	Weighted Estimates of Functions in Generalized Vanishing Morrey Spaces	662
13.2.2	Weighted Hardy Operators in Generalized Vanishing Morrey Spaces	664
13.3	Maximal, Potential and Singular Integral Operators in Generalized Variable Exponent Morrey Spaces	670
13.3.1	Preliminaries: Estimates of Norms of Truncated Potentials	670
13.3.2	Variable Exponent Generalized Morrey Spaces: Definitions and Statements of the Main Results	672
13.3.3	Proofs of Theorems 13.37–13.38, 13.39–13.40 and 13.42–13.46	677
13.3.4	Corollaries	685
13.4	Sublinear Operators in Variable Exponent Herz Spaces	686
13.4.1	Introduction	686
13.4.2	Preliminaries on Herz Spaces with Constant Exponents	687
13.4.3	Herz Spaces with Variable Exponent $p(x), q(t), \alpha(t)$	688
13.4.4	An Auxiliary Lemma	692
13.4.5	Boundedness of Sublinear Operators of Singular-type in Herz Spaces	693
13.4.6	Sobolev-type Theorems for the Riesz Potential in Herz Spaces	700

13.5	Variable Exponent Morrey Spaces Defined on Quasimetric Measure Spaces	701
13.5.1	Preliminaries. Modified Morrey Spaces	701
13.5.2	Modified Maximal Function	703
13.5.3	Potentials. Boundedness	707
13.5.4	Regularity of Potentials	710
13.6	Maximal and Calderón–Zygmund Singular Operators on SHT	713
13.6.1	Maximal Functions on SHT	714
13.6.2	Singular Integrals	715
13.6.3	Applications to Singular Integrals on Fractal Sets	718
13.7	Variable Exponent Morrey–Campanato Spaces	719
13.7.1	Preliminaries	719
13.7.2	Variable Exponent Hölder Spaces	724
13.7.3	Variable Exponent Campanato Spaces	725
13.7.4	Embedding Theorem	727
13.7.5	Coincidence of Variable Exponent Campanato Spaces with Variable Exponent Morrey Spaces in the Case $\lambda_+ < 1$	728
13.7.6	Coincidence of Variable Exponent Campanato Spaces with Variable Exponent Hölder Spaces in the Case $\lambda_- > 1$	732
13.8	Comments to Chapter 13	736
14	Singular Integrals and Potentials in Grand Lebesgue Spaces	
14.1	Maximal Functions and Hilbert Transform on the Interval $(0, 1)$	743
14.1.1	Hardy–Littlewood Maximal Operator and Hilbert Transform	744
14.1.2	Constants in One-weight Inequalities for \mathcal{M} and H	754
14.1.3	Integral Operators on SHT: Constants in One-weight Inequalities	757
14.2	Criteria Under B_p Conditions	762
14.2.1	General-type Theorem	767
14.2.2	Hardy Operators	770
14.2.3	Applications to Fractional Integrals and Maximal Functions	773
14.3	Sobolev-type Theorem	775
14.3.1	Fractional Integrals and Fractional Maximal Functions in Unweighted Grand Lebesgue Spaces	775
14.3.2	Sobolev Embedding in Weighted Generalized Grand Lebesgue Spaces	777
14.3.3	One-sided Potentials	782

- 14.4 One-weight Sobolev-type Theorem for Potentials with Product Kernels 784
 - 14.4.1 Boundedness and Unboundedness of the Operators T^α and \mathcal{M}_α^S in the Unweighted Case 785
 - 14.4.2 Multiple Potentials in Weighted Grand Lebesgue Spaces 788
 - 14.4.3 Potentials and Fractional Maximal Operators in $WL_w^{p,\theta}(X, \mu)$ and $L_w^{p,\theta}(X, \mu)$ Spaces 793
 - 14.4.4 Multiple Riemann–Liouville and Weyl Transforms in the Spaces $L^{p,\theta}([0, 1]^2)$ 795
 - 14.4.5 Further Remarks 802
- 14.5 Trace Problem for Potentials on Quasimetric Measure Spaces 804
 - 14.5.1 General Result 804
 - 14.5.2 Fractional Maximal Operators and Potentials 806
- 14.6 Trace Problem for One-sided Potentials 811
- 14.7 Strong Maximal Functions 813
- 14.8 Potentials on the Upper Half-space 815
 - 14.8.1 Fefferman–Stein-type Inequality 817
- 14.9 Calderón–Zygmund Inequality 819
- 14.10 Compactness in Grand Lebesgue Spaces 821
 - 14.10.1 A Kolmogorov–Riesz-type Theorem 824
- 14.11 Grand Variable Exponent Lebesgue Spaces 830
 - 14.11.1 Hardy–Littlewood Maximal Functions and Calderón–Zygmund Singular Integrals 833
 - 14.11.2 Fractional Integrals 844
- 14.12 Comments to Chapter 14 848
- 15 Grand Lebesgue Spaces on Sets of Infinite Measure**
- 15.1 Spaces $L_\nu^{p,\theta}(\Omega; \langle x \rangle^{-\lambda})$, $\Omega \subseteq \mathbb{R}^n$ 852
 - 15.1.1 Restrictions on the Choice of $\lambda(\varepsilon)$ for Embeddings of the Spaces $L_{\lambda(\varepsilon)}^{p-\varepsilon}(\Omega)$ 852
 - 15.1.2 Definition of Grand Lebesgue Spaces with Power Weights on Sets of Infinite Measure 855
- 15.2 Some Examples of Functions in $L_\nu^{p,\theta}(\mathbb{R}^n; \langle x \rangle^{-\lambda})$ 857
- 15.3 Grand Lebesgue Spaces on Sets of Infinite Measure: General Weights 859
- 15.4 Boundedness of Sublinear Operators in Weighted Grand Lebesgue Spaces 861
 - 15.4.1 A Transference Theorem 862
- 15.5 Another Viewpoint on Weighted Grand Lebesgue Spaces 864
 - 15.5.1 The Grand Spaces $\mathcal{L}_w^{p,\theta}(\Omega)$ 865

15.5.2	On the Boundedness of the Linear Operators in the Space $\mathcal{L}_{w,\beta}^p(\Omega)$	866
15.6	On a More General Approach	867
15.7	Comments to Chapter 15	870
16	Fractional and Singular Integrals in Grand Morrey Spaces	
16.1	Generalized Grand Morrey Spaces	871
16.2	Reduction Lemma	874
16.3	Maximal Operator	877
16.4	Singular Integrals	881
16.5	Riesz-type Potentials	884
16.6	Modified Maximal Functions	891
16.7	Potentials on an SHT	893
16.8	Fractional Integrals on Non-homogeneous Spaces	896
16.9	Commutators of Calderón–Zygmund Operators	901
16.10	Commutators of Potential Operators	905
16.11	Interior Estimates for Elliptic Equations	911
16.12	Elliptic Equations in Non-divergence Form	915
16.13	Comments to Chapter 16	924
17	Multivariable Operators on the Cone of Decreasing Functions	
17.1	Preliminaries	925
17.2	Riemann–Liouville Operators with Product Kernels	928
	17.2.1 Pointwise Estimates	929
	17.2.2 Two-weight Criteria	931
17.3	Riesz Potentials on the Cone of Non-increasing Functions	943
17.4	Potentials on \mathbb{R}_+	943
	17.4.1 Two-weight Criteria for Riesz Potentials with Product Kernels	949
17.5	Comments to Chapter 17	965
	Appendix: Grand Bochner Spaces	
A.1	Bochner Integral	967
A.2	Grand and Small Bochner–Lebesgue Spaces	968
A.3	Comments to the Appendix	973
	Bibliography	975
	Symbol Index	991
	Author Index	995
	Subject Index of Volume 2	999
	Subject Index of Volume 1	1001