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Editors

# Complex Geometry and Dynamics

The Abel Symposium 2013



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# Foreword

The Norwegian government established the Abel Prize in mathematics in 2002, and the first prize was awarded in 2003. In addition to honoring the great Norwegian mathematician Niels Henrik Abel by awarding an international prize for outstanding scientific work in the field of mathematics, the prize shall contribute toward raising the status of mathematics in society and stimulate the interest for science among school children and students. In keeping with this objective, the Niels Henrik Abel Board has decided to finance annual Abel Symposia. The topic of the Symposia may be selected broadly in the area of pure and applied mathematics. The Symposia should be at the highest international level and serve to build bridges between the national and international research communities. The Norwegian Mathematical Society is responsible for the events. It has also been decided that the contributions from these Symposia should be presented in a series of proceedings, and Springer Verlag has enthusiastically agreed to publish the series. The Niels Henrik Abel Board is confident that the series will be a valuable contribution to the mathematical literature.

Helge Holden  
Chair of the Niels Henrik Abel Board  
Trondheim, Norway



# Preface

The theme of the Abel Symposium 2013 was *Complex Geometry*, and it was held at the Norwegian University of Science and Technology, Trondheim, during July 2–5. The event attracted 43 participants and featured presentations by 22 speakers. The scientific agenda primarily focused on geometric problems in Several Complex Variables and Complex Dynamics, including holomorphic laminations/foliations, the  $\bar{\partial}$ -equation, CR-geometry, pluripotential theory, and function theory. The aim of the Abel Symposium was to present the state of the art on these topics and to discuss future research directions. The speakers and titles were:

1. Bedford, E. *Automorphisms of blowups of projective space*
2. Berndtsson, B. *The openness problem and complex Brunn-Minkowski inequalities*
3. Błocki, Z. *Hörmander's  $\bar{\partial}$ -estimate, some generalizations and new applications*
4. Demailly, J.-P. *On the cohomology of pseudoeffective line bundles*
5. Dihn, T.-C. *Positive closed  $(p, p)$ -currents and applications in complex dynamics*
6. Ebenfelt, P. *Partial rigidity of degenerate CR-embeddings into spheres*
7. Fornstnerič, F. *Complex analysis and the Calabi-Yau problem*
8. Grushevsky, S. *Meromorphic differentials with real periods and the geometry of the moduli space of Riemann surfaces*
9. Huang, X. *Analyticity of the local hull of holomorphy for a codimension two real-submanifold in  $\mathbb{C}^n$*
10. Kohn, J. *Weakly pseudoconvex CR manifolds*
11. McMullen, C. *Entropy and dynamics on complex surfaces*
12. Merker, J. *Siu-Yeung holomorphic sections of  $\text{Sym}^m T_X^*$*
13. Mok, N. *On the Zariski closure of an infinite number of totally geodesic subvarieties of  $\Omega/\Gamma$*
14. Nemirovski, S. *Topology and several complex variables*
15. Ohsawa, T. *Levi flats in Hopf surfaces*
16. Sibony, N. *Dynamics of foliations by Riemann surfaces*
17. Stensønes, B. *Real analytic domains and plurisubharmonic functions*
18. Ueda, T. *Semi-parabolic fixed points and their bifurcations in complex dimension 2*
19. Yau, S.-T. *Period integrals, counting curves, and mirror symmetry*

20. Yau, S. *Nonconstant CR morphisms between compact strongly pseudoconvex CR manifolds and etale covering between resolutions of isolated singularities*
21. Yeung, S.-K. *Complex hyperbolicity on the moduli of some higher-dimensional manifolds*
22. Zhou, X. *Some results on  $L^2$ -extension problem with optimal estimate*

The scientific committee consisted of John Erik Fornæss (NTNU), Marius Irgens (NTNU), Yum-Tong Siu (Harvard), Erlend F. Wold (Oslo), and Shing-Tung Yau (Harvard).

During the symposium, a dinner was held in honor of Yum-Tong Siu's 70th birthday. We would like to dedicate these proceedings to him.

The participants at the symposium were:

Eric Bedford	Ngaiming Mok
Bo Berndtsson	Stefan Nemirovski
Zbigniew Błocki	Takeo Ohsawa
Fusheng Deng	Tron Omland
Jean-Pierre Demailly	Marius Overholt
Klas Diederich	Nils Øvrelid
Tien-Cuong Dinh	Alexander Rashkovskii
Peter Ebenfelt	Nessim Sibony
John Erik Fornæss	Yum-Tong Siu
Franc Forstnerič	Berit Stensønes
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Joseph Kohn	Sai-Kee Yeung
Erik Løw	Jian-Hua Zheng
Benedikt Magnusson	Xiang-Yu Zhou
Curtis McMullen	Minxian Zhu
Joël Merker	

We would like to thank the Norwegian Mathematical Society and the Nils Henrik Abel Memorial Fund for giving us the opportunity to host the Abel Symposium. We would also like to thank the administration at NTNU for their great help with the organizing and Ruth Allewelt at Springer for her help with preparing the proceedings.

Trondheim, Norway  
 Trondheim, Norway  
 Oslo, Norway  
 May 15, 2015

John Erik Fornæss  
 Marius Irgens  
 Erlend Fornæss Wold



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