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Introduction to Sofic and Hyperlinear Groups and Connes' Embedding Conjecture

With an Appendix by Vladimir Pestov

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Preface

Analogy is one of the most effective techniques of human reasoning: When we face new problems, we compare them with simpler and already known ones, in the attempt to use what we know about the latter ones to solve the former ones. This strategy is particularly common in Mathematics, which offers several examples of abstract and seemingly intractable objects: Subsets of the plane can be enormously complicated but, as soon as they can be approximated by rectangles, then they can be measured; Uniformly finite metric spaces can be difficult to describe and understand but, as soon as they can be approximated by Hilbert spaces, then they can be proved to satisfy the coarse Novikov's and Baum-Connes's conjectures.

These notes deal with two particular instances of such a strategy: Sofic and hyperlinear groups are in fact the countable discrete groups that can be approximated in a suitable sense by finite symmetric groups and groups of unitary matrices. These notions, introduced by Gromov and Rădulescu, respectively, at the end of the 1990s, turned out to be very deep and fruitful, and stimulated in the last 15 years an impressive amount of research touching several seemingly distant areas of mathematics including geometric group theory, operator algebras, dynamical systems, graph theory, and more recently even quantum information theory. Several long-standing conjectures that are still open for arbitrary groups were settled in the case of sofic or hyperlinear groups. These achievements aroused the interest of an increasing number of researchers into some fundamental questions about the nature of these approximation properties. Many of such problems are to this day still open such as, outstandingly: Is there any countable discrete group that is not sofic or hyperlinear? A similar pattern can be found in the study of II_1 factors. In this case, the famous conjecture due to Connes (commonly known as Connes' embedding conjecture) that any II_1 factor can be approximated in a suitable sense by matrix algebras inspired several breakthroughs in the understanding of II_1 factors, and stands out today as one of the major open problems in the field.

The aim of this monograph is to present in a uniform and accessible way some cornerstone results in the study of sofic and hyperlinear groups and Connes' embedding conjecture. These notions, as well as the proofs of many results, are here presented in the framework of model theory for metric structures. We believe that this point of view, even though rarely explicitly adopted in the literature, can contribute to a better understanding of the ideas therein, as well as provide additional tools to attack many remaining open problems. The presentation is nonetheless self-contained and accessible to any student or researcher with a graduate-level mathematical background. In particular, no specific knowledge of logic or model theory is required.

Chapter 1 presents the conjectures and open problems that will serve as common thread and motivation for the rest of the survey: Connes' embedding conjecture, Gottschalk's conjecture, and Kaplansky's conjecture. Chapter 2 introduces sofic and hyperlinear groups, as well as the general notion of metric approximation property; outlines the proofs of Kaplansky's direct finiteness conjecture and the algebraic eigenvalues conjecture for sofic groups; and develops the theory of entropy for sofic group actions, yielding a proof of Gottschalk's surjunctivity conjecture in the sofic case. Chapter 3 discusses the relationship between hyperlinear groups and the Connes' embedding conjecture; establishes several equivalent reformulations of the Connes' embedding conjecture due to Haagerup-Winsløw and Kirchberg; describes the purely algebraic approach initiated by Rădulescu and carried over by Klep-Schweighofer and Juschenko-Popovich; and finally outlines the theory of Brown's invariants for II_1 factors satisfying the Connes' embedding conjecture. An appendix by V. Pestov provides a pedagogically new introduction to the concepts of ultrafilters, ultralimits, and ultraproducts for those mathematicians who are not familiar with them, and aiming to make these concepts appear very natural.

The choice of topics is unavoidably not exhaustive. A more detailed introduction to the basic results about sofic and hyperlinear groups can be found in [125, 126]. The surveys [118, 120, 121] contain several other equivalent reformulations of the Connes' embedding conjecture in purely algebraic or C*-algebraic terms.

This survey originated from a short intensive course that the authors gave at the Universidade Federal de Santa Catarina in 2013 in occasion of the "Workshop on sofic and hyperlinear groups and the Connes' embedding conjecture" supported by CAPES (Brazil) through the program "Science without borders", PVE project 085/2012. We would like to gratefully thank CAPES for its support, as well as the organizers of the workshop Daniel Gonçalves and Vladimir Pestov for their kind hospitality, and for their constant and passionate encouragement.

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