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Discontinuous Galerkin Method

Analysis and Applications to Compressible Flow

Springer

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Preface

Many real-world problems are described by partial differential equations whose numerical solution represents an important part of numerical mathematics. There are several techniques for their solution: the finite difference method, the finite element method, spectral methods and the finite volume method. All these methods have advantages as well as disadvantages. The first three techniques are suitable particularly for problems in which the exact solution is sufficiently regular. The presence of interior and boundary layers appearing in solutions of singularly perturbed problems (e.g., convection-diffusion problems with dominating convection) or discontinuities in solutions of nonlinear hyperbolic equations lead to some difficulties. On the other hand, finite volume techniques based on discontinuous, piecewise constant approximations are very useful in solving convection-diffusion problems, but their disadvantage is their low order of accuracy.

The most recent technique for the numerical solution of partial differential equations is the *discontinuous Galerkin method* (DGM), which uses ideas of both the finite element and finite volume methods. The DGM is based on piecewise polynomial but discontinuous approximations, which provides robust numerical processes and high-order accurate solutions.

During the past two decades the DGM has become very popular and a number of works has been concerned with its analysis and applications. It appeared that the DGM is suitable for the numerical solution of a number of problems for which other techniques fail or have difficulties. We can mention singularly perturbed problems with boundary and internal layers, which exist in solutions of convection-diffusion equations with dominating convection.

Another possibility represents problems with solutions containing discontinuities and steep gradients, as in the case of nonlinear hyperbolic problems and compressible flow. This means that the DGM is suitable for the numerical solution of problems appearing particularly in fluid dynamics, hydrology, heat and mass transfer and environmental protection on the one hand, but also financial mathematics and image processing on the other hand. Moreover, the DGM offers considerable flexibility in the choice of the mesh design; indeed, the DGM easily handles non-matching and non-uniform grids, even anisotropic, with different polynomial approximation degrees on different elements. This allows for a simple treatment of *hp*-variants of adaptive techniques. Finally, the DGM can easily be parallelized, which is demanding in complex numerical simulations.

This book is devoted to the theory and applications of the discontinuous Galerkin method. The first part of this book deals with theoretical aspects of the discontinuous Galerkin (DG) method applied to the numerical solution of scalar nonlinear convection-diffusion problems. Scalar equations serve as models for several applications treated in the second part of the book. Our aim is to present the DG discretization of model problems and to derive (a priori) error estimates. Theoretical results are supported by numerical experiments demonstrating the accuracy of the DG methods.

In order to better understand the basic principles of the discontinuous Galerkin method, we start from a numerical solution of the simple Poisson problem having mixed Dirichlet–Neumann boundary conditions. Hence, in Chaps. 2 and 3, we describe the DG discretization and the derivation of error estimates in detail in order to familiarize non-specialist readers with theoretical tools used in the DGM. We tried to have material self-contained as much as possible. Therefore, these chapters contain similar material on the DGM as other monographs.

In Chaps. 4–6 the main attention is paid to the analysis of discontinuous Galerkin techniques for solving nonstationary, nonlinear convection-diffusion problems. Chapter 7 is devoted to some generalizations of the DGM: the *hp*-version of the DGM, the use of general polygonal elements and the effect of numerical integration. Theoretical results are demonstrated by the solution of numerous test problems.

The second part (formed by Chaps. 8–10) deals with applications of the DGM to solving gas dynamics problems. The numerical schemes, proposed and analyzed in the first part of the book, are extended to solving the system of equations describing compressible flow, namely, in Chap. 8, the compressible Euler equations are solved, Chap. 9 is devoted to the solution of viscous flow described by the compressible Navier–Stokes equations, and in Chap. 10 the DGM is applied to simulating compressible flow in time-dependent domains and to the interaction of compressible flow with elastic structures. We also discuss the numerical solution of the resulting systems of algebraic equations which is a fundamental aspect in the practical use of the DGM for solving industrial problems. The treatment in the last three chapters is accompanied by test problems and also technically relevant applications proving the flexibility, accuracy and robustness of the described discontinuous Galerkin schemes.

We hope that the book will be useful to specialists—namely, pure and applied mathematicians, aerodynamists, engineers, physicists and natural scientists. We also expect that the book will be suitable for graduate and postgraduate students in mathematics and in the technical sciences.

As for references, there is a rapidly increasing amount of literature on theoretical aspects and applications of the DGM. We tried to quote the works relevant to the topics of the book, but it is clear that many significant references have been unintentionally omitted. We apologize in advance to those authors whose contributions are not mentioned or do not receive the attention they deserve. We have tried to avoid errors, but some may remain. Readers are welcome to send any correction electronically to the address dolejsi@karlin.mff.cuni.cz or feist@karlin.mff.cuni.cz.

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