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Linking Game-Theoretical Approaches with Constructive Type Theory

Dialogical Strategies, CTT Demonstrations
and the Axiom of Choice

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To Göran Sundholm and Gerhard Heinzmann

Preface

A brief examination of the most recent literature in logic shows that a host of research in this area studies the interface between games, logic, and epistemology. These studies provide the basis for ongoing enquiries in the history and philosophy of logic, going from the Indian, the Greek, the Arabic traditions, the Obligations of the Middle Ages to the most contemporary developments in the fields of theoretical computer science, computational linguistics, artificial intelligence, social sciences, and legal reasoning. In fact, a dynamic turn, as Johan van Benthem puts it, is taking place where the epistemic aspects of inference are linked with game theoretical approaches to meaning.¹

This turn came about in the 1960s when Paul Lorenzen and Kuno Lorenz developed dialogical logic—inspired by Wittgenstein’s language games and mathematical game theory—and when some time later Jaakko Hintikka combined game-theoretical semantics with epistemic (modal) logic. If we had to pinpoint a date, it would be 1958 with Lorenzen’s talk² “Logik und Agon.”

However, these two approaches to logic—the dialogical one and the one based on Hintikka’s GTS—springing from a dynamic reading of the epistemic conception of logic, disregarded a major advance precisely in the epistemic approach to logic, namely the development by Per Martin-Löf of Constructive Type Theory (CTT)—with the sole exception of the pioneering paper of Ranta (1988).³ This framework,

¹New results in linear logic by J.-Y. Girard at the interface between mathematical game theory and proof theory on the one hand and argumentation theory and logic on the other resulted in the work of, among others, S. Abramsky, J. van Benthem, A. Blass, H. van Ditmarsch, D. Gabbay, M. Hyland, W. Hodges, G. Japaridze, E. Krabbe, H. Prakken, G. Sandu, D. Walton and J. Woods, who placed game semantics at the center of a new concept of logic in which logic is understood as a dynamic instrument of inference. In this context see also Blass (1992), Abramsky and Mellies (1999), Girard (1999), Lecomte and Quatrini (2010, 2011), Lecomte (2011), Lecomte and Tronçon (2011).

²Published as Lorenzen (1960).

³We will discuss further on the differences with our own approach, however, let us point out that Ranta’s paper was written from the CTT perspective on the game theoretical approach to meaning, rather than the other way round.

providing a type theoretical development of the Curry-Howard-isomorphism and introducing dependent-types, leads to the formulation of a fully-interpreted language—a language with content challenging the standard metalogical approach to meaning of model theoretic semantics in general and of the modal-interpretation of epistemic logic in particular. Furthermore, an inferential and contentual language based on CTT (Sundholm 1986, 2001; Ranta 1994) has now been successfully applied not only to the semantics of natural languages but also to the foundations of logic, computer sciences, and constructive mathematics. Philosophically speaking, CTT shares the Kantian view that judgements, rather than propositions, constitute the foundation of knowledge. According to this perspective, the basic ontology is determined by the two fundamental forms of judgement, namely categorical judgements with *independent* proof-objects and hypothetical judgements with *dependent* proof-objects (i.e., functions). See Chap. 1.

Interestingly enough CTT supplies, as discussed by Rahman and Clerbout (2013), the basis for research on the same areas that the dialogical framework—after some initial analyses—stopped to work on, namely the foundations of mathematics and the development of a general dialogical theory of meaning. Up to now, the lack of interface between the game theoretical approaches and CTT is particularly striking because of their common philosophical ground. Let us very briefly expound on this. One way to put it is to follow Mathieu Marion's⁴ suggestion to use Brandom's (1994, 2000) pragmatist take on inferentialism which is led by three insights, two of Kantian origin on the one hand and one stemming from Brandom's reading of Hegel on the other hand:

- that judgements are the fundamental units of knowledge,
- that human cognition and action are characterized by certain kinds of normative assessment,⁵ and

⁴In fact, Marion (2006, 2009, 2010) was the first to propose a link between Brandom's pragmatist inferentialism and dialogue logic in the context of Wilfried Hodges challenging to the game-theoretical approaches (2001, 2004). Another relevant antecedent of the present work is the Ph.D. dissertation of Keiff (2006–2009) providing a thorough formulation of the dialogical approach within the framework of speech-act theory.

⁵The normative aspect, established on the shift from *Cartesian certainty* to *bindingness of rules* distinguishes Brandom's pragmatism from others':

One of the strategies that guided this work is a commitment to the fruitfulness of shifting theoretical attention from the Cartesian concern with the grip we have on concepts—for Descartes, in the particular form of the centrality of the notion of **certainty** (...)—to the Kantian concern with the grip concepts have on us, that is, the notion of **necessity** as the bindingness of the rules (including inferential ones) that determine how it is correct to apply those concepts. (Brandom 1994, p. 636)

- that communication is mainly conceived as cooperation in a joint social activity rather than as sharing contents.⁶

As mentioned above, the key aspect of the epistemic approach is that assertion or judgement amounts to a knowledge claim, independently of classical or intuitionistic views—cf. Prawitz (2012, p. 47). So if an expression’s meaning stems from its role in assertions, then the approach to meaning is epistemic. According to Brandom, the normative aspect is implemented through W. Sellar’s notion of games of giving and asking for reasons, games that deploy the way commitments and entitlements are intertwined. Indeed, according to Brandom’s view, it is the chain of commitments and entitlements in a game of giving and asking for reasons that binds up judgement with inference.⁷ Sundholm (2013) provides the following formulation of the notion of inference in a communicative context which can be seen as a core description of Brandom’s pragmatist inferentialism:⁸

When I say “Therefore” **I give others my authority for asserting the conclusion, given theirs for asserting the premisses.**

This is quite close to the main tenet of the dialogical approach to meaning, though with one crucial difference. The pragmatist approach to meaning of the dialogical framework not only endorses Brandom’s pragmatist inferentialism claim that the meaning of linguistic expressions is related to their role in games of questions and answers, but it also endorses Brandom’s notion of justification of a judgement as involving the interaction of commitments and entitlements. The important difference is that dialogicians insist that more fundamental lower levels should be distinguished—as discussed in Chap. 2. These lower semantic levels include (i) the description of how to formulate a suitable question for a given posit and how to answer it, and (ii) the development of plays, composed of several combinations of sequences of questions and answers brought forward as responses to the posit of a thesis. From the dialogical perspective, the level of judgements corresponds to the final stage of the chain of interactions just mentioned. More

⁶In relation to the model of holistic communication which is considered here, Brandom writes:

Holism about inferential significances has different theoretical consequences depending on whether one thinks of communication in terms of **sharing** a relation to one and the same **thing** (grasping a common meaning) or in terms of **cooperating** in a joint **activity** (...)
(Brandom 1994, p. 479).

⁷Moreover, according to Brandom, games of asking for reasons and giving them constitute the basis of any linguistic practice:

Sentences are expressions whose unembedded utterance performs a speech act such as making a claim, asking a question, or giving a command. Without expressions of this category, there can be no speech acts of any kind, and hence no specifically linguistic practice. (Brandom 2000, p. 125)

⁸Actually, Sundholm bases his formulation on J.L. Austin’s remarks in his celebrated paper “Other Minds” (Austin 1946) rather than on Brandom’s work.

precisely, the justifications of judgements correspond to the level of winning strategies, selecting those plays that turn out to be relevant for the drawing of inferences. Furthermore, as our discussion of the Axiom of Choice (Chap. 4) shows, the game theoretical take on dependent types is rooted on choice dependences, which can be seen as a result of intertwining games of questions and answers.

Let us point out that the distinctions drawn within the dialogical framework between local meaning, play level and strategy level, which we will explain in detail in Chap. 2, seem to provide an answer to Brandom's problem that the "grasp of concepts" amounts to the mastery of inferential roles but this

(...) does not mean that in order to count as grasping a particular concept an individual must be disposed to make or otherwise endorse in practice all the right inferences involving it. To be in the game at all, one must make **enough** of the right moves—but how much is enough is quite flexible. (Brandom 1994, p. 636)

Indeed, from the dialogical point of view, in order to grasp the meaning of an expression, the individual does not need to know the moves ensuring his victory (he must not have a winning strategy) and does not even need to win at all. What is required is that he knows what the relevant moves are which he is entitled and committed to (local meaning) in order to develop a play. In a similar way, knowing how to play chess does not necessarily mean to actually be in possession of a winning strategy. Knowing how to play allows to know what can count as a winning strategy, when there is one: strategic legitimacy (*Geltung*) is not to be found at the level of meaning-explanation. Thus, one way to see the motivations for proposing to link CTT and games is to give the technical elements binding the pragmatist approach that grasps concepts in Brandom's style with the proof-theoretical CTT take on meaning.

At this point of the discussion, we hope that the grounds for working out systematically the links between game theoretical approaches and CTT—or at least a glimpse of them—should be clear enough. A pending task, that we will not undertake here, is to discuss how the rigorous elaboration of a fully-interpreted language in terms of CTT fits with Brandom's pragmatic inferentialism. However, as stressed in Chap. 4, a fully-interpreted language lies at the core of the dialogical take on meaning.⁹ More generally and summing up, the development of a dialogical approach to Constructive Type Theory can be motivated by the following considerations:

1. In his book *The Interactive Stance*, Ginzburg (2012) stresses the utmost importance of taking conversational (interactive) aspects into account in order to develop a theory of meaning, in which meaning is constituted during the interaction. In order to implement such a theory of meaning Ginzburg uses CTT in which the so-called "metalogical" rules constituting meaning are explicitly imported into the object language. Moreover, Ginzburg designed some kind of

⁹For a thorough discussion on the subject see Rahman and Clerbout (2013).

language games called dialogical-gameboards in order to capture the dynamic aspects of everyday dialogues.

Now, if we take seriously the claim that meaning is constituted by and within interaction, then we expect that the semantics of the underlying logical elements is also understood dialogically. In this context, a dialogical approach to Constructive Type Theory provides both a dialogical framework for the underlying logic and a natural link to the dialogical gameboards.

2. In some recent work, Dybjer (2012) proposed to study the relation between the intensional and the extensional notions of identity, defined in CTT by game theoretical means. Once more, in this context an approach to CTT that is already game-theoretical seems to be desirable.
3. The links between game-theoretical approaches and CTT seem to be very natural. Indeed, the CTT approach is very natural to the dialogical framework in which the meaning of the logical constants is given by moves such as challenges and choices constituting the explicit development of a play. Challenges and choices are in fact part of the object language of the development of a play.
4. The fathers of the dialogical framework had the project of developing both a general theory of meaning (and not only of logic) and to open a new path on the foundations of (constructive) mathematics and logic. Here, too, it is CTT that enriches the dialogical approach.¹⁰

Given these motivations, it is important to prove that the dialogical approach to CTT developed by the two authors in recent papers¹¹ yields a notion of winning strategy which really corresponds to the notion of CTT demonstration. This is the main aim of the present study. More precisely, we will consider on the one hand Martin L of's Constructive Type Theory,¹² and, on the other hand, dialogues with play-objects¹³ on the other hand. Our purpose is to show the following:

There is a winning **P**-strategy in the dialogical game for φ if and only if there is a CTT demonstration for $p : \varphi$, where p is some proof-object for φ .

One important aspect of the present study is that we restrict ourselves to the logically valid fragment of CTT. The reason is that, once our goal for that fragment is achieved we can extend the result to cover the whole CTT system.

Now, because demonstrations in Martin L of's CTT are given in the lines of a natural deduction calculus, our demonstration of the equivalence result will be based on Rahman et al. (2009) in which the connection between dialogical semantics and natural deduction has been investigated. However, the present study does not assume knowledge of the papers on dialogues and CTT mentioned afore. There are anyway various significant adjustments that must be made, namely:

¹⁰In fact there are already some recent applications of the interaction of the dialogical approach and CTT in the context of legal reasoning (Rahman 2014).

¹¹Rahman and Clerbout (2013), Rahman et al. (2014).

¹²Cf. Martin-L of (1984), Nordstr om et al. (1990), Ranta (1994).

¹³We explain further on the notion of play-objects.

- In the aforementioned paper, the Fitch-style notation has been adopted. In the present study we will not follow that notation and this requires adapting the translation procedure between strategies and CTT demonstrations to a formulation much closer to Gentzen's.
- The paper by Rahman et al. (2009) deals mainly with standard classical propositional logic. Thus, in the context of the present study we need to extend the result to quantifiers and develop an explicit language featuring play-objects for the play level, and proof-objects for the strategical level which yields the CTT notion of proof-object.
- The paper on natural deduction does not provide a systematic method of extracting demonstrations from winning strategies. Such a systematic study has been carried out by Clerbout (2014a, b) in relation to Tableaux. In the present study (one direction of) the method of Clerbout will be adapted to the aims specified above.

This study is thus organized as follows.

Chapters 1 and 2 recall the two frameworks at stake, insisting in particular on the dialogical framework. For one thing, it is true that the dialogical approach is not very widespread in the literature. In addition, the particular system we work with is very recent and thus uncommon even for readers familiar with the dialogical approach. Thus, Chap. 2 contains both the standard account of dialogical games (for readers unfamiliar with it) and the enriched version at stake (for the demonstration of the equivalence result). Our overview of the basics of CTT is briefer mostly because there is a well-established and detailed literature on this approach. We also believe that it is sufficient for the purpose of investigating and showing the connections between the two approaches. However, we refer readers who are not familiar with the CTT framework to detailed introductions such as Martin-Löf (1984), Nordström et al. (1990), and Granström (2011).

Chapter 3 is devoted to the demonstration of the left-to-right direction of the equivalence result.

In Chap. 4 we illustrate the use of the algorithm of Chap. 3 by showing how to transform a specific winning strategy into a CTT demonstration of the Axiom of Choice. The chapter insists on the intensional version of the axiom, but the extensional version is also considered.

Chapter 5 develops the algorithm from CTT demonstrations to dialogical strategies. Some readers may wonder why the material on the Axiom of Choice is placed between the two directions of the demonstration of the equivalence result. The reason is that it relies only on the first direction, covered in Chap. 3, and can thus be understood without the second direction. Actually it could be confusing to place it after proving the second direction in which it plays no part. Hence our choice to place it as an illustration of the procedure developed in Chap. 3, and a kind of interlude before resuming the demonstration of the equivalence result.

We conclude by introducing elements of discussion which are to be developed in subsequent papers. These papers will involve the analysis of the notion of truth rather than the validity and development of a new tableaux system for CTT that follows from the notion of *core* developed in the present study.

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Lille
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Contents

1	Brief Reminder of Constructive Type Theory	1
1.1	Fundamentals of the CTT Approach	1
1.2	The Basic CTT Framework for Intuitionistic Predicate Logic	5
1.2.1	About the Rules	5
1.2.2	Intuitionistic Logic in CTT	6
	References	8
2	Dialogues with Play-Objects	9
2.1	Standard Dialogical Games	11
2.2	The Formation of Propositions	16
2.3	Play-Objects	21
2.4	The Development of a Play	31
2.5	Example	34
	References	36
3	From Dialogical Strategies to CTT Demonstrations	39
3.1	Towards the Core	39
3.1.1	Getting Rid of Infinite Ramifications	39
3.1.2	Disregarding Formation Rules	42
3.1.3	Disregarding Irrelevant Variations in the Order of O -moves	42
3.2	From the Core to a CTT Demonstration	43
3.2.1	Generalities	44
3.2.2	The Algorithm	46
3.3	Adequacy of the Translation Algorithm	49
	References	52

4 The Dialogical Take on the Axiom of Choice, and Its Translation into CTT 53

4.1 The Dialogical Take on the Axiom of Choice 54

 4.1.1 Two Plays on the Axiom of Choice 56

 4.1.2 The Core of a Winning P-strategy 59

4.2 From the Dialogical Strategy to the CTT Demonstration of the Axiom of Choice 61

4.3 The Extensional Version of the Axiom of Choice 77

 4.3.1 Martin-Löf on the Extensional Version with Uniqueness 79

 4.3.2 The Dialogical Way 80

References 85

5 Building a Winning P-strategy Out of a CTT Demonstration 87

5.1 Transforming CTT Demonstrations 88

 5.1.1 Guidelines 88

 5.1.2 The Procedure 91

5.2 Adequacy of the Algorithm 94

References 96

6 Conclusions and Work in Progress 97

References 99

Errata to: Linking Game-Theoretical Approaches with Constructive Type Theory E1

List of Figures

Figure 4.1	Core of the winning P -strategy	60
Figure 4.2	From the core to the demonstration (Initial stage)	61
Figure 4.3	From the core to the demonstration (Stage 2)	62
Figure 4.4	From the core to the demonstration (Stage 3.1)	62
Figure 4.5	From the core to the demonstration (Stage 3.2)	63
Figure 4.6	From the core to the demonstration (Stage 3.3)	63
Figure 4.7	From the core to the demonstration (Stage 3.4)	64
Figure 4.8	From the core to the demonstration (Stage 3.5)	64
Figure 4.9	From the core to the demonstration (Stage 3.6)	65
Figure 4.10	From the core to the demonstration (Stage 4.1)	65
Figure 4.11	From the core to the demonstration (Stage 4.2)	66
Figure 4.12	From the core to the demonstration (Stage 4.3.1)	67
Figure 4.13	From the core to the demonstration (Stage 4.3.2)	67
Figure 4.14	From the core to the demonstration (Stage 4.3.3)	68
Figure 4.15	From the core to the demonstration (Stage 4.3.4)	68
Figure 4.16	From the core to the demonstration (Stage 4.3.5)	69
Figure 4.17	From the core to the demonstration (Stage 4.3.6)	70
Figure 4.18	From the core to the demonstration (Stage 4.3.7)	70
Figure 4.19	From the core to the demonstration (Stage 4.3.8)	71
Figure 4.20	From the core to the demonstration (Stage 4.4.1)	71
Figure 4.21	From the core to the demonstration (Stage 4.4.2)	72
Figure 4.22	From the core to the demonstration (Stage 4.4.3)	72
Figure 4.23	From the core to the demonstration (Stage 4.4.4)	73
Figure 4.24	From the core to the demonstration: after going through stages 1 to 4.4.4	74
Figure 4.25	From the core to the demonstration: after Stage 5 (partial view)	75
Figure 4.26	The CTT demonstration obtained from Fig. 4.1 after stages 1 to 6 of Sect. 4.2.	78

List of Tables

Table 2.1	Particle rules in standard dialogical games	11
Table 2.2	Formation rules	17
Table 2.3	Example of a formation dialogue 1	19
Table 2.4	Example of a formation dialogue 2	20
Table 2.5	Particle rules	22
Table 2.6	A dialogue involving the donkey sentence	35
Table 4.1	First play for the axiom of choice	57
Table 4.2	Second play for the axiom of choice	58
Table 4.3	Play for the extensionality of f	83