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A C_p -Theory Problem Book

Compactness in Function Spaces

Springer

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Preface

This is the third volume of the series of books of problems in C_p -theory entitled “A C_p -Theory Problem Book”, i.e., this book is a continuation of the two volumes subtitled *Topological and Function Spaces* and *Special Features of Function Spaces*. The series was conceived as an introduction to C_p -theory with the hope that each volume could also be used as a reference guide for specialists.

The first volume provides a self-contained introduction to general topology and C_p -theory and contains some highly non-trivial state-of-the-art results. For example, Section 1.4 presents Shapirovsky’s theorem on the existence of a point-countable π -base in any compact space of countable tightness and Section 1.5 brings the reader to the frontier of the modern knowledge about realcompactness in the context of function spaces.

The second volume covers a wide variety of topics in C_p -theory and general topology at the professional level, bringing the reader to the frontiers of modern research. It presents, among other things, a self-contained introduction to Advanced Set Theory and Descriptive Set Theory, providing a basis for working with most popular axioms independent of ZFC.

This present volume basically deals with compactness and its generalizations in the context of function spaces. It continues dealing with topology and C_p -theory at a professional level. The main objective is to develop from scratch the theory of compact spaces most used in Functional Analysis, i.e., Corson compacta, Eberlein compacta, and Gul’ko compacta.

In Section 1.1 of Chapter 1, we build up the necessary background presenting the basic results on spaces $C_p(X)$ when X has a compact-like property. In this section, the reader will find the classical theorem of Grothendieck, a very deep theorem of Reznichenko on ω -monolithity, under $\text{MA} + \neg\text{CH}$, of a compact space X if $C_p(X)$ is Lindelöf, as well as the results of Okunev and Tamano on non-productivity of the Lindelöf property in spaces $C_p(X)$.

The main material of this volume is placed in Sections 1.2–1.4 of Chapter 1. Here we undertake a reasonably complete and up-to-date development of the theory of Corson, Gul’ko, and Eberlein compacta. Section 1.5 develops the theory of splittable

spaces and gives far-reaching applications of extension operators in both C_p -theory and general topology.

We use all topological methods developed in the first two volumes, so we refer to their problems and solutions when necessary. Of course, the author did his best to keep *every* solution as independent as possible, so a short argument could be repeated several times in different places.

The author wants to emphasize that if a postgraduate student mastered the material of the first two volumes, it will be more than sufficient to understand every problem and solution of this book. However, for a concrete topic much less might be needed. Finally, let me outline some points which show the potential usefulness of the present work.

- the only background needed is some knowledge of set theory and real numbers; any reasonable course in calculus covers everything needed to understand this book;
- the student can learn all of general topology required without recurring to any textbook or papers; the amount of general topology is strictly minimal and is presented in such a way that the student works with the spaces $C_p(X)$ from the very beginning;
- what is said in the previous paragraph is true as well if a mathematician working outside of topology (in functional analysis, for example) wants to use results or methods of C_p -theory; he (or she) will find them easily in a concentrated form or with full proofs if there is such a need;
- the material we present here is up to date and brings the reader to the frontier of knowledge in a reasonable number of important areas of C_p -theory;
- this book seems to be the first self-contained introduction to C_p -theory. Although there is an excellent textbook written by Arhangel'skii (1992a), it heavily depends on the reader's good knowledge of general topology.

Mexico City, Mexico

Vladimir V. Tkachuk

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Introduction

The term “ C_p -theory” was invented to abbreviate the phrase “The theory of function spaces endowed with the topology of pointwise convergence”. The credit for the creation of C_p -theory must undoubtedly be given to Alexander Vladimirovich Arhangel’skii. The author is proud to say that Arhangel’skii also was the person who taught him general topology and directed his PhD thesis. Arhangel’skii was the first to understand the need to unify and classify a bulk of heterogeneous results from topological algebra, functional analysis, and general topology. He was the first to obtain crucial results that made this unification possible. He was also the first to formulate a critical mass of open problems which showed this theory’s huge potential for development.

Later, many mathematicians worked hard to give C_p -theory the elegance and beauty it boasts nowadays. The author hopes that the work he presents for the reader’s judgement will help to attract more people to this area of mathematics.

The main text of this volume consists of 500 statements formulated as problems; it constitutes Chapter 1. These statements provide a gradual development of many popular topics of C_p -theory to bring the reader to the frontier of the present-day knowledge. A complete solution is given to every problem of the main text.

The material of Chapter 1 is divided into five sections with 100 problems in each one. The sections start with an introductory part where the definitions and concepts to be used are given. The introductory part of any section *never exceeds two pages and covers everything that was not defined previously*. Whenever possible, we try to save the reader the effort of ploughing through various sections, chapters, and volumes, so we give the relevant definitions in the current section not caring much about possible repetitions.

Chapter 1 ends with some bibliographical notes to give the most important references related to its results. The selection of references is made according to the author’s preferences and by no means can be considered complete. However, a complete list of contributors to the material of Chapter 1 can be found in our bibliography of 400 items. It is my pleasant duty to acknowledge that I consulted the paper of Arhangel’skii (1998a) to include quite a few of its 375 references in my bibliography.

Sometimes, as we formulate a problem, we use without reference definitions and constructions introduced in other problems. The general rule is to try to find the relevant definition *not more than ten problems before*.

The first section of Chapter 1 deals with the spaces $C_p(X)$ for a compact X . It includes the classical theorem of Grothendieck, a deep result of Reznichenko on ω -monolithity, under $MA + \neg CH$, of any compact space X such that $C_p(X)$ is Lindelöf, as well as the examples of Okunev and Tamano on non-productivity of the Lindelöf property in function spaces.

Sections 1.2–1.4 of Chapter 1 represent the core of this volume; they constitute a development of the theory of Corson, Gul’ko, and Eberlein compacta. These classes of compact spaces are of utmost importance not only in topology but also in functional analysis. These sections include all classical results (characterizations via families of retractions, categorical and topological properties of these compact spaces, etc.) together with comparatively new material on Sokolov spaces and a radical generalization of a cornerstone result of Gul’ko on the Corson property of any compact space X whose $C_p(X)$ is Lindelöf Σ .

It is worth mentioning that Section 1.3 features a detailed description and proof of the properties of a famous example of Reznichenko of a Talagrand compact space X such that $X = \beta(X \setminus \{a\})$ for some point $a \in X$. This example has never been published by the author; since it disproves quite a few conjectures in both topology and functional analysis, the respective paper circulated as a preprint for more than 20 years before the example was described in a monograph on Functional Analysis.

Every Section of Chapter 1 has numerous topics that are developed up to the frontier of the present-day knowledge. In particular, Section 1.2 introduces the reader to the technique of adequate sets for constructing Corson compact spaces, Section 1.3 contains the results on domination of $C_p(X)$ by second countable spaces, and Section 1.4 gives a Gruenhage’s characterization of Eberlein compactness of X in terms of a covering property of $(X \times X) \setminus \Delta$.

The complete solutions of all problems of Chapter 1 are given in Chapter 2. Chapter 3 begins with a selection of 100 statements which were proved as auxiliary facts in the solutions of the problems of the main text. This material is split into 8 sections to classify the respective results and make them easier to find. Chapter 4 consists of 100 open problems presented in 8 sections with the same idea: to classify this bulk of problems and make the reader’s work easier.

Chapter 4 also witnesses an essential difference between the organization of our text and the book of Arhangel’skii and Ponomarev (1974): *we never put unsolved problems in the main text as is done in their book*. All problems formulated in Chapter 1 are given complete solutions in Chapter 2, and the unsolved ones are presented in Chapter 4.

There is little to explain about how to use this book as a reference guide. In this case, the methodology is not that important and the only thing the reader wants is to find the results he (or she) needs as fast as possible. To help with this, the titles of chapters and sections give the first approximation. To better see the material of a chapter, one can consult the second part of the Contents section where a detailed

summary is given; it is supposed to cover all topics presented in each section. Besides, the index can also be used to find necessary material.

To sum up the main text, I believe that the coverage of C_p -theory will be reasonably complete and many of the topics can be used by postgraduate students who want to specialize in C_p -theory. Formally, this book can also be used as an introduction to general topology. However, it would be a somewhat biased introduction, because the emphasis is always given to C_p -spaces and the topics are only developed when they have some applications in C_p -theory.

To conclude, let me quote an old saying which states that the best way for one to learn a theorem is to prove it oneself. This text provides a possibility to do this. If the reader's wish is to read the proofs, there they are concentrated immediately after the main text.