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An Invitation to Web Geometry

Springer

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IMPA Monographs
ISBN 978-3-319-14561-7 ISBN 978-3-319-14562-4 (eBook)
DOI 10.1007/978-3-319-14562-4

Library of Congress Control Number: 2015930015

Mathematics Subject Classification (2010): 53A60, 53A20, 14H

Springer Cham Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media
(www.springer.com)

Para Dayse e Jorge,
os meus Pastores.
J.V.P.

Pour Min' et Nin'
L.P.

Preface

The first purpose of this text was to serve as supporting material for a minicourse on web geometry delivered at the 27th Brazilian Mathematical Colloquium, which took place at IMPA in the last week of July 2009. Still, it contains much more than what can possibly be covered in five lectures of 1 h each. The abundance of material is due to the second purpose of this text: to convey some of the beauty of web geometry and provide an account, as self-contained as possible, of some of the exciting advancements the field has witnessed in the last few years.

We have tried to make this text as little demanding as possible in terms of prerequisites.

It is true that familiarity with the basic language of algebraic and complex geometry is sometimes welcome but, except in very few passages, not much more is needed. An effort has been made to explain, even if sometimes superficially, every unusual concept appearing in the text.

Contents of the Chapters

The table of contents tells rather precisely what the book is about. The following descriptions give additional information.

Chapter 1 is introductory and describes the basic notions of web geometry. Most of the content of this chapter is well known. A notable exception is the notion of duality for global webs on projective spaces $\mathbb{P}^n$, which appears to be new when $n > 2$. This notion is discussed in Sect. 1.4.3. The first two sections, more specifically Sects. 1.1 and 1.2, are of rather elementary nature and might be read by an undergraduate student.

Chapter 2 is about the notions of abelian relation and rank. It offers an outline of Abel's method to determine the abelian relations of a given planar web. It also gives a description of the abelian relations of planar webs admitting an infinitesimal symmetry. The most important results in this chapter are Chern's bound on the rank (Theorem 2.3.8) and the normal form for the conormals of a web of maximal rank

(Proposition 2.4.10). This last result is proved using a geometric approach based on classical concepts and results from projective algebraic geometry, which are described in detail.

Chapter 3 is devoted to Abel's notorious addition theorem. It first deals with the case of smooth projective curves, then tackles the general case after introducing the notion of abelian differential. Section 3.3 gives a rather precise description of the Castelnuovo curves (projective curves of maximal genus) hence of some algebraic webs of maximal rank. Section 3.4 expounds new results: an (easy) variant of Abel's theorem (Proposition 3.4.1), which is combined with Chern's bound on the rank so as to obtain bounds on the genus of curves included in abelian varieties (*cf.* Theorem 3.4.5).

Chapter 4 is where the converse to Abel's theorem is demonstrated. Its proof is given through a reduction to the planar case, which is then treated using a classical argument that can be traced back to Darboux. Then follows a presentation of some algebraization results. Important concepts such as Poincaré's and canonical maps for webs are discussed in this chapter. Our only contribution is of formal nature and is situated in Sect. 4.3, where we endeavor to work as intrinsically as possible.

Chapter 5 is entirely devoted to Trépreau's algebraization result. The proof that is presented is essentially the same as the original one [130]. The only "novelty" in this chapter is Sect. 5.1.2, where a geometric interpretation of the proof is given. As in the preceding chapter, an effort was made to formulate some of the results and their proofs as intrinsically as possible.

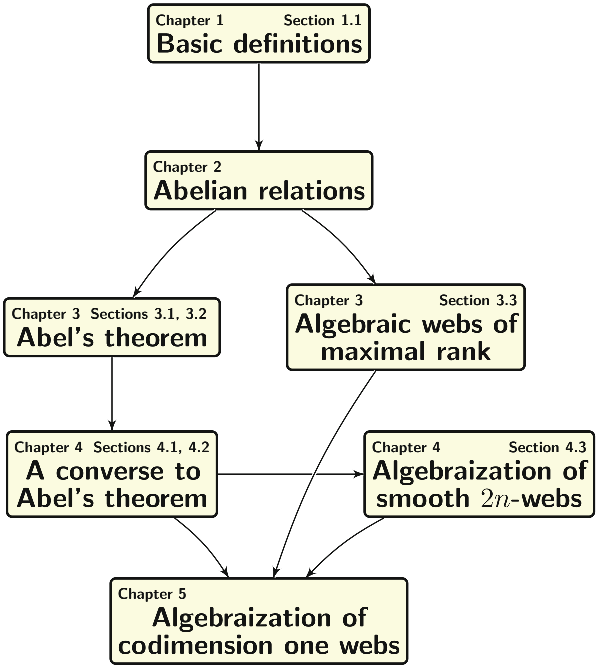
Chapter 6 takes up the case of planar webs of maximal rank, more specifically of exceptional planar webs. Classical criteria that characterize linearizable webs on the one hand and maximal rank webs on the other hand are explained. Then the existence of exceptional planar k-webs, for arbitrary $k \geq 5$, is established through the study of webs admitting infinitesimal automorphisms. The classification of the so-called CDQL webs on compact complex surfaces obtained recently by the authors is also reviewed. The chapter ends with a brief discussion about all the examples of planar exceptional webs we are aware of.

Finally, in the **Appendix**, the reader will find historical notes on the development of the field from its origins to recent advances.

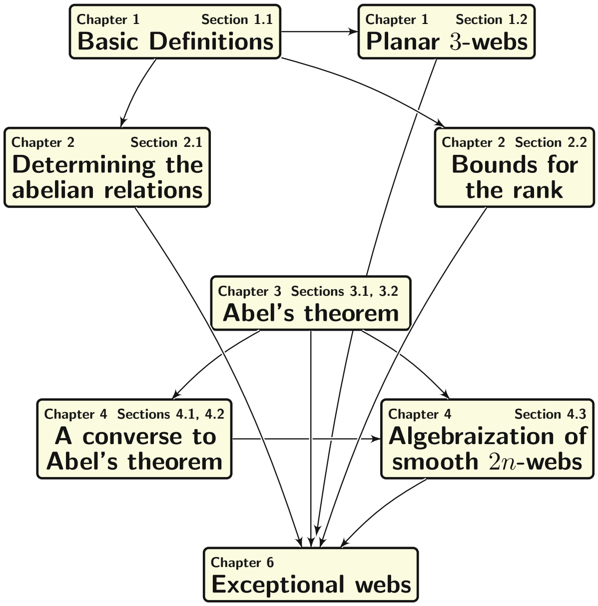
How to Use This Book

The logical organization of this book is rather simple: the reader with enough time to spare can read it from cover to cover.

Those who are mostly interested in Bol–Trépreau's algebraization theorem may find useful the minimal route suggested by the graph below.



Those who are anxious to learn more about exceptional webs might prefer to use instead the following graph as a reading guide.



This book would have taken much longer to come to light without the invitation of Márcio Gomes Soares to submit a minicourse proposal to the 27th Brazilian Mathematical Colloquium. Besides Soares, we would like to thank Hernan Maycol Falla Luza and Paulo Sad, who caught a number of misprints and mistakes appearing in preliminary versions.

We are also indebted to Annie Bruter for her many corrections as well as for her help in translating into English a draft of the appendix originally written in French.

Last but not least, Jorge wishes to thank Dayse and Luc wishes to thank Mina for their patience and unconditional support during the writing of this book.

Rio de Janeiro, Brazil — Jorge Vitório Pereira
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Conventions

All the **definitions**, including this one, are presented in bold case and have a corresponding entry at the index.

Unless stated otherwise all the geometric entities like curves, surfaces, varieties, and manifolds considered in this text are reduced and complex holomorphic. Curves, surfaces, and varieties may be singular and may have several irreducible components. The manifolds are smooth connected varieties.

Web geometry lies on the interface of local differential geometry and projective algebraic geometry. Throughout the text, the reader will be confronted with both local non-algebraic subvarieties of the projective space and global, and algebraic and compact, projective subvarieties. A projective curve, surface, variety, or manifold will mean a compact curve, surface, variety, or manifold contained in some projective space. Beware that some authors use the term projective to qualify any subvariety, compact or not, algebraic or not, of a given projective space.

Throughout the text, there will be references to points $x \in (\mathbb{C}^n, 0)$ and properties of germs at the point x. The latter has to be understood as a point in a sufficiently small neighborhood of the origin and the property as a property of some representative of the germ defined in this very same sufficiently small neighborhood.

Below is a list of some of the notations used in this book.

- If n is a positive integer, $\underline{n}$ will stand for the set $\{1, \ldots, n\}$;
- For any $q \in \mathbb{N}$, $\mathbb{C}_q[x_1, \ldots, x_n]$ will stand for the vector space of degree q homogeneous polynomials in $x_1, \ldots, x_n$;
- The span of a subset S of a projective space or of a vector space will be denoted by $\langle S \rangle$;
- If f is a differentiable function depending on some variables $x_1, \ldots, x_n$, we will sometimes denote by f_{x_i} the partial derivative $\partial f / \partial x_i$.