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Jacobi Forms, Finite Quadratic Modules and Weil Representations over Number Fields

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To my beloved father, Mustafa Boylan

Foreword

The current research monograph presents a breakthrough in at least three ways. Firstly, it introduces the simple but striking idea that the index of a Jacobi form over a totally real number field should be a lattice of rank one over the ring of integers rather than a number. The classical theory of Jacobi forms over the rational numbers uses positive integers for the index. Accordingly the first attempts to extend this theory to number fields tried to define the index by totally positive numbers in the different of the field. It soon turned out that this is too restrictive to obtain a satisfactory theory. However, if one views the index of classical Jacobi forms as (one half of) the Gram matrix of a rank one lattice over the rational integers, it becomes clear why, for general number fields, scalar indices will not suffice to capture all Jacobi forms. Indeed, as soon as the ring of integers of a given number field is no longer a principal ideal domain, we lose the one-to-one correspondence between lattices of rank one and numbers. The missing right notion of index blocked for a long time the research on Jacobi forms over number fields. As shown in this monograph, the consequent use of lattices as indices leads finally to a smooth and consistent theory.

Secondly, the development of a theory of Jacobi forms over number fields was also blocked by the lack of concrete or interesting examples. Again this monograph breaks this spell. It shows that there are indeed interesting examples. More precisely, it gives us in the last chapter a complete description of all Jacobi forms of singular weight on the full Hilbert modular group over any given totally real number field. These new functions generalize to number fields the classical Jacobi theta function which occurs in the Jacobi triple product, and which is essentially equal to the Weierstrass sigma function. One might expect that Boylan's theta functions will play a similar important role in the theory of Hilbert modular forms, the theory of abelian varieties or algebraic number theory as the classical Jacobi theta function, or equivalently, the Weierstrass sigma function.

Finally, indispensable for the study of Jacobi forms over the rationals is their realization as vector-valued modular forms. These modular forms take values in Weil representations. Again the theory of Jacobi forms over number fields was blocked since the corresponding objects for number fields were only vaguely known as part of Weil's general and abstract theory of what is known nowadays as Weil representations. For concrete considerations the Weil representations of Hilbert modular groups were apparently considered as too complicated. Again this research monograph surprises by showing that this is not at all true. It develops from scratch an appealing complete theory of finite quadratic modules and their associated Weil representations over arbitrary (not necessarily totally real) number fields. It describes the decomposition into irreducible parts of those Weil representations which are important for the Jacobi forms considered in this monograph. This theory alone provides already a valuable and indispensable tool for future research not only on Jacobi forms but also for the representation theory of Hilbert modular groups.

The theory of Jacobi forms over number fields is far from being at a stage as the corresponding theory over the rationals. However, the beauty of this theory already glimpses through if one looks at its basics and the concrete examples as they are presented in this book. This monograph will serve well as the cornerstone for building up a complete arithmetic theory of the newly introduced Jacobi forms. Indeed, there is currently already various work in progress.

In [SS14] the authors calculate the dimension of the spaces of vector-valued Hilbert modular forms with special emphasis on deriving explicit formulas for the dimensions of spaces of Jacobi forms over number fields of weight greater than 2. The approach is based on a general Eichler–Selberg trace formula for vector-valued Hilbert modular forms, and on the theory of Weil representations and their connection to Jacobi forms as developed in this monograph. The article [BHS14] determines the structure of the ring of Jacobi forms over $\mathbb{Q}(\sqrt{5})$ as module over the ring of Hilbert modular forms. The research project [SW14] aims to develop the theory of Hecke operators for Jacobi forms over number fields, and to study the connection of this new type of Jacobi forms to Siegel–Hilbert modular forms. The article [Boy14] calculates the Fourier coefficients of Jacobi Eisenstein series over number fields and gives thereby the first concrete examples for a deep arithmetic connection between the new Jacobi forms and Hilbert modular forms. In [BS14b] the authors plan to summarize the results of these (and possibly other) research activities, to give further explicit examples of liftings from Jacobi forms over number fields to Hilbert modular forms, and, in particular, to present a complete corresponding Hecke theory.

The main interest for constructing a theory for Jacobi forms over number fields arose from the fact that we expect several deep results from the theory of elliptic modular forms and Jacobi forms over $\mathbb{Q}$ to hold true for the number field case too. In particular, we expect liftings from Jacobi forms over number fields to Hilbert modular forms. Moreover, the Fourier coefficients of the Jacobi forms should encode the vanishing at the critical point of twisted L-functions associated with Hilbert modular forms. This is, in particular, interesting in the context of a generalized Birch and Swinnerton–Dyer conjecture for elliptic curves over number fields.

This monograph is an important step towards such a theory. It opens the door to a new and fascinating world of so far unseen functions. I hope that it will stimulate more researchers to follow this invitation to a new exciting subject.

Bonn, Germany Nils-Peter Skoruppa
September 2014

Preface

In analogy to the theory of classical Jacobi forms which has proven to have various important applications ranging from number theory to physics, we develop in this research monograph a theory of Jacobi forms over arbitrary totally real number fields. However, we concentrate here mainly on the connection of such Jacobi forms and the theory of Weil representations, leaving out important topics like Hecke theory and liftings to Hilbert modular forms, which still have to be developed. We hope to come back to those topics in later publications, but that the present work stimulates already further interest in this rich new theory. Here, we develop, first of all, a theory of finite quadratic modules over number fields and their associated Weil representations. Next we develop in detail the basics of the theory of Jacobi forms over number fields and the connection to Weil representations. As a main application of our theory, we are able to describe explicitly all singular Jacobi forms over arbitrary totally real number fields whose indices have rank one. We expect that these singular Jacobi forms play a similar important role in this newly founded theory of Jacobi forms over number fields as the Weierstrass sigma function does in the classical theory of Jacobi forms.

I thank the Max-Planck Institute for Mathematics in Bonn for its hospitality and the beautiful research environment which they provided during the year 2013 when I was working on finalizing this book and on related topics. I also thank İstanbul Üniversitesi for allowing me to spend the year 2013 in Bonn. I thank the Mathematical Sciences Research Institute in Berkeley for hosting me in spring 2011 at a still early stage of this research project, and Tongji University in Shanghai and Chennai Mathematical Institute for giving me the possibility to lecture in fall 2009 on various very early results on Jacobi forms over number fields. Finally, I thank Nils-Peter Skoruppa for having me introduced to this fascinating subject, for showing constantly interest in my work, and for being always ready to share his profound vision of mathematics. Last but not least, I thank all those who cannot all be named here explicitly without omissions but have been influential for my career.

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Hatice Boylan

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Introduction

A Jacobi form of weight k and index m (both half integral) on the full modular group $\mathrm{SL}(2,\mathbb{Z})$ is a holomorphic function $\phi(\tau,z)$ on the product $\mathbb{H}\times\mathbb{C}$ of the complex upper half plane $\mathbb{H}$ with the set of complex numbers $\mathbb{C}$ such that $\psi(\tau,x,y) := \phi(\tau,x\tau+y)\,e^{2\pi i m x^2\tau}$ satisfies the following properties:

(i) The function $\psi(\tau,x,y)$ is quasi-periodic in the real variables x and y with period 1.
(ii) For fixed rational x,y, the map $\tau\mapsto\psi(\tau,x,y)$ defines an elliptic modular form of weight k (possibly with character) on the principal congruence subgroup $\Gamma(a)$ of $\mathrm{SL}(2,\mathbb{Z})$, where a denotes the square of the least common multiple of the denominators of x and y.

The first property implies that, for fixed τ, the map $z\mapsto\phi(\tau,z)$ defines a theta function (a holomorphic section of a line bundle) on the elliptic curve $\mathbb{C}\big/\big(\mathbb{Z}\tau+\mathbb{Z}\big)$. If we study n-dimensional abelian varieties whose endomorphism ring contains the ring of integers $\mathcal{O}$ of a totally real number field K of degree n over $\mathbb{Q}$, then we find naturally analogs of Jacobi forms. We will call them Jacobi forms over the number field K. A careful analysis shows, however, that we have to replace the index m by a totally positive definite integral $\mathcal{O}$-lattice of rank one. Such a lattice can always be represented by a pair $(\mathfrak{c},\omega)$, where $\mathfrak{c}$ is a fractional $\mathcal{O}$-ideal and ω a totally positive element in K such that $\mathfrak{c}^2\omega$ is contained in the inverse different of K (see Proposition 3.10). If K is the field of rational numbers and $\mathcal{O}$ is the ring of integers $\mathbb{Z}$, then such a lattice can always be represented by a pair $(\mathbb{Z},2m)$, i.e. by the $\mathbb{Z}$-module $\mathbb{Z}$ equipped with the $\mathbb{Z}$-bilinear form $(x,y)\mapsto 2mxy$, where m is a positive integer. The main difference here is that, for a number field K of class number greater than 1, a finitely generated, torsion free $\mathcal{O}$-module is not in general isomorphic to $\mathcal{O}$, but only to a fractional $\mathcal{O}$-ideal, whose ideal class might be not trivial.

A Jacobi form over K of half integral weight k and index $\underline{L} = (\mathfrak{c},\omega)$ is a holomorphic function $\phi(\tau,z)$ on $\mathbb{H}^n\times\mathbb{C}^n$ such that the function $\psi(\tau,x,y) := \phi(\tau,x\tau+y)\,e^{2\pi i\,\mathrm{tr}(\frac{1}{2}M(\omega)x^2\tau)}$ satisfies:

(i) The function $\psi(\tau, x, y)$ is quasi-periodic in the variables x and y in $\mathbb{R}^n$ with respect to the $\mathscr{O}$-sublattice $M(\mathfrak{c})$.
(ii) For fixed x and y in $M(K)$, the map $\tau \mapsto \psi(\tau, x, y)$ defines a Hilbert modular form of weight k (possibly with character) on the principal congruence subgroup $\Gamma(\mathfrak{a})$ of $\mathrm{SL}(2, \mathscr{O})$, where $\mathfrak{a}$ is the square of the least common multiple of the denominators of $a\mathfrak{c}^{-1}$ and of $b\mathfrak{c}^{-1}$ with $x = M(a)$ and $y = M(b)$.

Here M denotes the Minkowski embedding of K into $\mathbb{R}^n$, which maps a to the vector whose j-th component equals $\sigma_j(a)$, where we use a fixed enumeration σ_1, $\ldots$, σ_n of the embeddings of K into $\mathbb{R}$. Moreover, when writing $x\tau + y$ or $M(\omega)x^2\tau$, we view $\mathbb{C}^n$ as a ring with respect to component-wise multiplication. Finally, $\mathrm{tr}(z)$, for z in $\mathbb{C}^n$, denotes the sum of the components of z.

Note that the first property expresses the fact that, for fixed τ, the map $z \mapsto \phi(\tau, z)$ defines a theta function (a holomorphic section in a line bundle) on the abelian variety $\mathbb{C}^n \big/ \big(M(\mathscr{O})\tau + M(\mathscr{O})\big)$. For a precise definition of Jacobi forms over number fields we refer the reader to Definition 3.45. A justification of the informal description given here can be found in the Appendix of Chap. 3. Later, it will also be more convenient to use $\mathbb{C} \otimes_{\mathbb{Q}} K$ instead of $\mathbb{C}^n$ since the first object carries naturally several algebraic structures which we shall make use of, and it allows for coordinate independent calculations.

One of the first steps into an interesting theory of Jacobi forms is, of course, to exhibit explicit examples. As it turns out, for number fields different from $\mathbb{Q}$, it is in fact already not trivial and challenging to construct examples. In this monograph, after developing a sufficiently general theory of Jacobi forms over number fields, we determine explicitly all *singular Jacobi forms over number fields*, i.e. all Jacobi forms over number fields whose weight equals $1/2$ (see Definition 3.47 and Proposition 4.1).

The singular Jacobi forms over $\mathbb{Q}$ have been determined by Skoruppa in [Sko85, p. 27]. Namely, for $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$, set

$$\vartheta(\tau, z) := \sum_{s \in \mathbb{Z}} \left(\tfrac{-4}{s}\right) q^{s^2/8} \zeta^{s/2} \quad \left(q^n(\tau) := e^{2\pi i n\tau}, \zeta^n(z) := e^{2\pi i s z}\right).$$

(Here $\left(\frac{-4}{s}\right)$ denotes the nontrivial Dirichlet character modulo 4). The function ϑ is a Jacobi form over $\mathbb{Q}$ on the full modular group of weight $1/2$ and index $1/2$. In particular, ϑ is a singular Jacobi form. Skoruppa [Sko85, p. 27] showed that $\vartheta(\tau, dz)$ and $\vartheta^*(\tau, dz)$, where

$$\vartheta^*(\tau, z) := \frac{\vartheta(\tau, 2z)}{\vartheta(\tau, z)} \eta(z) = \sum_{s \in \mathbb{Z}} \left(\tfrac{12}{s}\right) q^{s^2/24} \zeta^{s/2},$$

and where d is a positive integer, are the only singular Jacobi forms over $\mathbb{Q}$ on the full modular group.

What makes the singular Jacobi forms interesting is that they occur in various important areas of mathematics. First of all, $\vartheta(\tau, z)$ is, up to normalization, the

Weierstrass' sigma-function $\sigma(\tau, z)$ associated with the elliptic curve $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$. Namely, we have

$$\vartheta(\tau, z) = \eta(\tau)^3 e^{z^2 q \frac{d}{dq} \log \eta(\tau)} \sigma(\tau, z),$$

where $\eta(\tau) = q^{1/24} \prod_{n \geq 1}(1 - q^n)$ is the Dedekind's eta function. As such $\vartheta(\tau, z)$ is the basic functions out of which can be constructed all theta functions on elliptic curves. In the arithmetic theory of elliptic curves, it shows up as the Green's function for the elliptic curve $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$. Moreover, $\vartheta(\tau, z)$ and $\vartheta^*(\tau, z)$ show up in the theory of Kac–Moody algebras via the famous triple and quintuple identity, respectively. For example, the Jacobi triple product identity

$$\vartheta(\tau, z) = q^{1/8}(\zeta^{1/2} - \zeta^{-1/2}) \prod_{n \geq 1}(1 - q^n)(1 - q^n \zeta)(1 - q^n \zeta^{-1})$$

can be interpreted as the Weyl–Kac denominator identity for a certain affine Kac–Moody algebra.

In view of the indicated importance of the function $\vartheta(\tau, z)$, it is natural to ask whether such functions exist also for the abelian varieties $\mathbb{C}^n/(M(\mathcal{O})\tau + M(\mathcal{O}))$ mentioned above. It is then also natural to expect that they are also singular Jacobi forms, which explains our interest in determining all singular Jacobi forms over number fields.

We explain our main results concerning singular Jacobi forms (see Theorems 4.2 and 4.3 for more precise statements).

Theorem *There exist nonzero singular Jacobi forms over K if and only if* 2 *splits completely in K and the principal genus of K contains an ideal of the form $\mathfrak{g}\mathfrak{d}^{-1}$, where $\mathfrak{g}$ is a (possibly empty) product of pairwise different prime ideals of degree* 1 *over* 3*, and where $\mathfrak{d}$ denotes the different of K.*

Recall that the principal genus of K is the set of fractional $\mathcal{O}$-ideals $\mathfrak{a}$ which represent a square in the narrow ideal class group $\mathrm{Cl}^+(K)$ of K, i.e. for which there exist a fractional $\mathcal{O}$-ideal $\mathfrak{c}$ and a totally positive ω in K such that $\mathfrak{a} = \mathfrak{c}^2\omega$. A theorem of Hecke [Hec81, Theorem 177] states that the different $\mathfrak{d}$ is a square in the ideal class group of K. However, it need not necessarily to be a square in the narrow ideal class group. A counterexample is provided by the number field $\mathbb{Q}(\sqrt{47})$. Note that 2 splits completely in this number field.

Theorem *Suppose* 2 *splits completely in K. If $\mathfrak{c}$ is a fractional $\mathcal{O}$-ideal and ω is a totally positive element in K such that $\mathfrak{g} := \mathfrak{c}^2\omega\mathfrak{d}$ is a (possibly empty) product of pairwise different prime ideals of degree* 1 *over* 3*, then*

$$\vartheta_{(\mathfrak{c},\omega)}(\tau, z) := \sum_{s \in \mathfrak{c}\mathfrak{g}^{-1}} \chi_{4\mathfrak{g}}(s') e^{2\pi i \operatorname{tr}\left(\frac{1}{8} M(\omega s^2)\tau\right)} e^{2\pi i \operatorname{tr}\left(\frac{1}{2} M(\omega s)z\right)}$$

defines a Jacobi form over K of singular weight $1/2$ and index $(\mathfrak{c}, \omega)$. Here $s' \in \mathscr{O}$ is such that $s \equiv s'\gamma \bmod 4\mathfrak{c}$, where $\gamma + 4\mathfrak{c}$ is a generator for $\mathfrak{c}\mathfrak{g}^{-1}/4\mathfrak{c}$. By $\chi_{4\mathfrak{g}}$, we denote the totally odd Dirichlet character modulo $4\mathfrak{g}$ (see Definition 2.44). Vice versa, every nonzero singular Jacobi form over K is (up to multiplication by a constant) of this form.

If $(\mathfrak{c}, \omega)$ is an index as in the theorem, then $\left(a^{-1}\mathfrak{c}, a^2\omega\right)$, for any nonzero a in K, is also such an index. Two indices are isomorphic if and only if one can be obtained from the other in this way, i.e. by multiplying with a suitable a (see Proposition 3.9). Note that the singular Jacobi forms associated with isomorphic lattices differ only in a trivial way. Namely, we have $\vartheta_{\left(a^{-1}\mathfrak{c}, a^2\omega\right)}(\tau, z) = \vartheta_{(\mathfrak{c},\omega)}(\tau, M(a)z)$. We shall see (Proposition 4.7) that the number of indices modulo isomorphism which admits a nonzero singular Jacobi form equals $|\,\mathrm{F}(K)| \cdot |\,\mathrm{Cl}^+(K)[2]|$, where $\mathrm{F}(K)$ is the subset of the principal genus consisting of ideals of the form $\mathfrak{g}\mathfrak{d}^{-1}$ with $\mathfrak{g}$ as in the last theorem, and where $\mathrm{Cl}^+(K)[2]$ is the kernel of the squaring map of the narrow ideal class group. For the field of rational numbers this number equals 2. The two classes of indices admitting a nonzero Jacobi form are represented by $(\mathbb{Z}, 1)$ and $(\mathbb{Z}, 3)$ and, indeed, we rediscover the forms from Skoruppa's theorem: $\vartheta_{(\mathbb{Z},1)} = \vartheta$ and $\vartheta_{(\mathbb{Z},3)} = \vartheta^*$.

We explain the other main themes of the book. In Chap. 3 we shall develop a general theory of Jacobi forms over number fields whose indices are arbitrary $\mathscr{O}$-lattices. In Chap. 4, we shall see that singular Jacobi forms correspond to one-dimensional submodules of certain (projective) $\mathrm{SL}(2, \mathscr{O})$-modules of theta functions (see Proposition 4.1) which turn out to be isomorphic to Weil representations associated with certain finite quadratic modules over number fields. A theory of finite quadratic modules over number fields and a theory of Weil representations associated with finite quadratic modules over number fields have not yet been worked out in the literature. Therefore we shall develop these theories in Chaps. 1 and 2, respectively. In Chap. 2, we decompose, in particular, the spaces of cyclic Weil representations into irreducible subrepresentations (see Theorem 2.5). This will give us the clue for determining explicitly all singular Jacobi forms whose indices are $\mathscr{O}$-lattices of rank one, since these correspond to the one-dimensional subrepresentations of cyclic Weil representations (see Theorem 2.6). Translating these results back to the language of Jacobi forms, we can then determine in Chap. 4 explicitly all singular Jacobi forms whose indices are $\mathscr{O}$-lattices of rank one. In the last section of this chapter we show how to construct explicitly Jacobi forms over number fields of non-singular weight, and we give examples. Finally, in the Appendix, we present tables concerning the first number fields which admit nonzero singular Jacobi forms.

Notations

In general, if the number field K is clear from the context, we often drop the subscript K, i.e. we write $\mathcal{O}$, $\mathfrak{d}$, $\mathrm{tr}(a)$, $\mathrm{N}(a)$, etc. for $\mathcal{O}_K$, $\mathfrak{d}_K$, $\mathrm{tr}_{K/\mathbb{Q}}(a)$, and $\mathrm{N}_{K/\mathbb{Q}}(a)$.

Let K be number field with ring of integers $\mathcal{O}$. A Dirichlet character modulo an integral $\mathcal{O}$-ideal $\mathfrak{a}$ is a map χ from $\mathcal{O}$ to $\mathbb{C}^*$ defined by

$$\chi(r) = \begin{cases} \chi'(r+\mathfrak{a}) & \text{if } (r,\mathfrak{a}) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where χ' is a group homomorphism from $(\mathcal{O}/\mathfrak{a})^*$ to $\mathbb{C}^*$.

An exact divisor $\mathfrak{b}$ of an integral $\mathcal{O}$-ideal $\mathfrak{a}$ is the ideal so that $\mathfrak{b} + \mathfrak{a}\mathfrak{b}^{-1} = \mathcal{O}$.

Let $\mathfrak{a}$ be a fractional $\mathcal{O}$-ideal and $\mathfrak{p}$ be a prime ideal of the number field K. We use $v_{\mathfrak{p}}(\mathfrak{a})$ for the valuation of $\mathfrak{a}$ at $\mathfrak{p}$, i.e. for the exponent of the exact power of $\mathfrak{p}$ occurring in the prime ideal factorization of $\mathfrak{a}$. Note that $v_{\mathfrak{p}}(\mathfrak{a})$ can be negative for some $\mathfrak{a}$. If $\mathfrak{a}$ is integral, we have $v_{\mathfrak{p}}(\mathfrak{a}) \geq 0$, for all $\mathfrak{p}$.

In expressions like $\sum_{\mathfrak{b}|\mathfrak{a}} \cdots$, where $\mathfrak{a}$ is an integral $\mathcal{O}$-ideal, it is always understood, if not otherwise stated, that $\mathfrak{b}$ runs through the integral $\mathcal{O}$-ideals dividing $\mathfrak{a}$. Similarly, in expressions like $\prod_{\mathfrak{p}|\mathfrak{a}} \cdots$ or $\prod_{\mathfrak{p}^a \| \mathfrak{a}} \cdots$ it is understood that $\mathfrak{p}$ runs through the prime ideals or exact prime ideal powers $\mathfrak{p}^a$ dividing $\mathfrak{a}$.

For a finite set M, the symbol $\mathbb{C}[M]$ stands for the $\mathbb{C}$-vector space of all functions from M into $\mathbb{C}$. A basis for this vector space is the set of all functions e_x ($x \in M$) such that $e_x(y)$ equals 1 or 0 accordingly $x = y$ or not.

Let H be a subgroup of finite index in the group G, and let ϕ be a complex-valued function on G which takes the same value on each coset of G/H. We use $\sum_{g \in G/H} \phi(g)$ as a short-hand notation for $\sum_{g \in R} \phi(g)$, where R is a complete set of representatives for G/H.

In the sequel theorems are numbered independently, whereas the numbering of lemmas, propositions, examples, and corollaries share the same numbering sequence.