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A Finite Element Primer for Beginners

The Basics

 Springer

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Preface

The purpose of this primer is to provide the reader with the basics of the Finite Element Method (FEM), primarily illustrated through a classical model problem, *linearized elasticity*. The topics covered are:

- Weighted residual methods and Galerkin approximations;
- A model problem for one-dimensional linear elastostatics;
- Weak formulations in one dimension;
- Minimum principles in one dimension;
- Error estimation in one dimension;
- Construction of Finite Element basis functions in one dimension;
- Gaussian quadrature;
- Iterative solvers and element-by-element data structures;
- A model problem for three-dimensional linear elastostatics;
- Weak formulations in three dimensions;
- Basic rules for element construction in three dimensions;
- Assembly of the system and solution schemes;
- An introduction to time-dependent problems and;
- An introduction to rapid computation based on domain decomposition and basic parallel processing.

The approach used herein is to introduce the basic concepts first in one dimension, then move on to three dimensions. A relatively informal style is adopted in this guide. These notes are intended as a “starting point,” which can be later augmented by the large array of rigorous, detailed books in the area of Finite Element analysis. Through teaching finite element classes for a number of years at UC Berkeley, my experience has been that the fundamental mathematics, such as vector calculus, linear algebra, and basic mechanics, exemplified by linearized elasticity, cause conceptual problems that impede the understanding of the Finite Element Method. Thus, appendices on these topics have been included. Finally, I am certain that, despite painstaking efforts, there remain errors of one sort or another. Therefore, I would be grateful if readers who find such flaws would contact me at zohdi@berkeley.edu.

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