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Mark V. Sapir

Combinatorial Algebra: Syntax and Semantics

With Contributions by
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Introduction

What Is This Book About

In analyzing proofs of results about various algebraic objects (groups, semigroups, rings), it is easy to notice two types of results: syntactic results involving words, automata, languages, and semantic results involving algebraic properties of subalgebras and homomorphic images, geometric properties of certain associated objects (graphs, manifolds, and more complicated metric spaces), dynamical properties of associated actions, etc. One of the goals of this book is to demonstrate deep connections between syntax and semantics and to show how syntax and semantics interact when we study fundamental questions concerning algebras. These include the Burnside-type questions (what makes an algebra finite?), the questions about growth (how large is an infinite algebra?), and the finite basis question (can this class of algebras be nicely described?).

The interaction between syntax and semantics is mutually beneficial. Sometimes syntax "helps" semantics. For instance, combinatorics of words and their 2-dimensional images—diagrams—helps solve Burnside-type problems and show that certain semigroups, groups and rings are infinite or finite. On the other hand semantic information about algebraic structures helps to prove whether they have or do not have finite bases of identities. Sometimes in order to prove syntactic results one needs to study semantic properties of associated objects. For example, in order to describe "avoidable identities" of semigroups, one needs semantic properties of subshifts associated with certain infinite sets of words.

What Is in the Book

The book has five chapters. The first chapter contains basic general definitions from algebra, language theory and symbolic dynamics that are used throughout the text. The main recurrent topics of this book—Burnside problems, growth of algebras, and the finite basis property—are also introduced.

The second and third chapters contain results about avoidable words and identities, including the description of semigroup varieties where the Burnside problem has a positive solution. Although the results are about words, the methods are semantic: with every infinite semigroup, one can associate a certain subshift, and recurrence properties of that subshift are used to establish algebraic properties of the semigroup. The third chapter contains Trahtman's recent (but already famous) proof of the old road coloring conjecture by Adler, Goodwyn and Weiss. The conjecture has its origin in dynamical systems but the proof is basically syntactic and belongs essentially to semigroup theory. The third chapter also contains applications of road coloring to the classification of subshifts of finite type.

The fourth chapter is about the Burnside and growth properties for associative rings, and a big part of the fifth chapter is devoted to the same properties for groups.

In particular, Chapter 4 includes short proofs of the celebrated Shirshov height theorem, the classical result by Dubnov, Ivanov, Nagata and Higman about local finiteness of rings satisfying nil identities, the Kruse–L'vov theorem about finite bases of identities of finite rings, and Belov-Kanel's counterexample for the Specht problem (that problem was one of the main problems that inspired the whole theory of varieties of rings). Gelfand–Kirillov dimension of associative algebras with polynomial growth is also considered there.

Chapter 5 begins with showing how to convert words that are equal to 1 in a group into 2-dimensional pictures—van Kampen diagram and back. Greendlinger's theorem about Dehn–Greendlinger's algorithm for small cancellation groups is proved next, which is followed by a discussion of various syntactic properties of hyperbolic groups. Grigorchuk's example of a group of intermediate growth (a solution of Milnor's problem) and the Bass–Guivarc'h computation of the growth functions of finitely generated nilpotent groups are also in that chapter. The chapter also includes, for the first time in the literature, a "road map" of a proof (due to Olshanskii) of one of the most important results in group theory: the Novikov–Adian theorem that there are infinite finitely generated groups of finite exponent. The main goal of the road map is to give the reader a relatively short and gentle introduction (the main ideas, methods, and "points of interest") to the very difficult original proof. Chapter 5 ends with a section about amenable groups. In particular, a solution (due to Adian) of the von Neuman–Day problem is explained there.

One of our goals in the book is to show that different algebraic objects have quite similar syntactic features and are strongly related. In order to do that we use several tools and objects as "recurring characters" throughout the book. For example, full binary trees are used in describing terms of free non-associative algebras, in one of the definitions of the R. Thompson group F, and in the proof that Grigorchuk's group has intermediate growth. Zimin words play a crucial role in studying Burnside properties of semigroups, in the definition of Baer radical in rings, etc. Zimin words even appeared, at least in spirit, in Olshanskii's proof of the Novikov–Adian theorem (Section 5.2.3) and as Zel'manov's words in one of the important applications of Zel'manov's solution of the restricted Burnside problem. Uniformly recurrent words (which have origin in symbolic dynamics) are used to study Burnside-type and finite basis properties of semigroups and inverse semigroups. Finite automata of several kinds as well as Church–Rosser rewriting systems also appear throughout the book.

This book is not only about results but also about methods. There are several "universal" methods used in many proofs in the book. For example, rewriting systems are often used to find canonical forms of words and other objects. The diversity of interconnected subjects discussed in the book is manifested in the fact that this is one of a very few books where both the ergodic theorem of George Birkhoff and the HSP theorem of Garrett Birkhoff are used (Theorems 3.9.14 and 1.4.27).

What Is Not in the Book

The main purpose of the book is to cover the foundations of combinatorial algebra and several important applications. The largest area that is barely touched on in the book is "Algorithmic problems in algebra", in particular the word problem. The reason is that the area is so large that it requires a separate book (the survey by Kharlampovich and Sapir [177] is 250 pages long and does not even mention the most important results in the area obtained during the last 15 years).

For Whom Is the Book Written

I have tried to make the book self-contained. Basically any undergraduate student who took standard linear algebra and abstract algebra classes can read this book. I taught courses based on this book for undergraduate as well as graduate students, and even a 6-week course on avoidable words (Chapters 1–3) for high school students in one of the Canada/USA Mathcamps. In short, no significant knowledge of mathematics is required. Nevertheless problem solving experience and certain mathematical maturity would definitely help in reading the book.

Exercises

The book contains more than 350 exercises. Some of them are easy, others are relatively hard. Although I do not provide solutions, harder exercises contain hints, which should help find solutions. In some cases I decided that it would be instructive to let the reader prove a theorem on his/her own, so the proof is divided into a series of exercises, each of which is not too difficult. Also quite often I formulate a statement that is almost, but not quite, obvious. After such a statement I write "why?" or "prove it!". These are little exercises, which help, I hope, understand the proofs better. These usually replace the phrases like "It is easy to see that…" and "By a straightforward computation we obtain…", which are often used in mathematical texts and which I find intimidating. The exercises "embedded" in the proofs in the book contain technical statements whose proofs can be skipped if the reader wants to learn the main ideas of the proofs as opposed to all the details of it. If I gave detailed proofs of all the exercises, the book would become twice or three times as long while containing essentially the same information.

Further Reading and Open Problems

Chapters 2–5 end with sections "Further reading and open problems". These are surveys of main recent results and open problems in the relevant areas. Each of the open problem formulated there can be a topic of a PhD thesis.

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