Euclides
Ab Omni Naevo Vindicatus:
Sive
Conatus Geometricus
Quo Stabiliuntur
Prima ipsa universae Geometriae Principia.

Auctore

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Euclid Vindicated from Every Blemish or A Geometric Endeavor in which are Established the Foundation Principles of Universal Geometry

Procemium ad lectorem

[IX] Quanta sit Elementorum Euclidis praestantia, ac dignitas, nemo omnium, qui Mathematicas disciplinas noverint, ignorare potest. Lectissimos hanc in rem testes adhibeo Archimedem, Apollonium, Theodosium, aliosque pene innumeros, ad haec usque nostra tempora rerum Mathematicarum Scriptores, qui non aliter haec Euclidis Elementa usurpant, nisi ut principia jam diu stabilita, ac penitus inconcussa. Verum tanta haec nominis celebritas vetare non potuit, quin multi ex Antiquis pariter, ac Recentioribus, iique Magni Geometrae naevos quosdam a se depraehensos censuerint in his ipsis pulcherrimis, nec unquam satis laudatis Elementis. Tres autem hujusmodi naevos designant, quos statim subnecto.

Primus respicit definitionem parallelarum, & sub ea Axioma, quod apud Clavium est decimumtertium Libri primi, ubi Euclides sic pronunciat: *Si in duas rectas lineas, in eodem plano existentes recta incidens linea duos ad easdem partes internos angulos minores duobus rectis cum eisdem efficiat, duae illae rectae lineae ad eas partes in infinitum protractae inter se mutuo incident*. Porro nemo est, qui dubitet de veritate expositi Pronunciati; sed in eo unice Euclidem accusant, quod nomine Axiomatis usus fuerit, quasi nempe ex solis terminis rite perspectis sibi ipsi faceret fidem. Inde autem non pauci (retenta caeteroquin Euclidaea parallelarum definitione) illius demonstrationem aggressi sunt ex iis solis Propositionibus Libri primi Euclidaei, quae praecedunt vigesimam nonam, ad quam scilicet usui esse incipit
[X] controversum Pronunciatum.

Sed rursum; quoniam antiquorum in hanc rem conatus visi non sunt adamussim scopum attingere; factum idcirco est, ut multi proximiorum temporum eximii Geometrae, idem pensum aggressi, necessariam censuerint novam quandam parallelarum definitionem. Itaque; cum Euclides parallelas rectas lineas definiat, *quae in eodem plano existentes, si in utramque partem in infinitum producantur, nunquam inter se mutuo incidunt*; postremis expositae definitionis vocibus has alias substituunt: *Semper inter se aequidistant*; adeo ut nempe singulae perpendiculares ab uno quolibet unius illarum puncto ad alteram demissae aequales inter se sint.

At nova rursum hinc oritur scissura. Nam aliqui, & ii sane acutiores, demonstrare conantur parallelas rectas lineas prout sic definitas, unde utique gradum faciant ad demonstrandum sub ipsis Euclidaeis vocibus controversum Pronunciatum, cui nimirum ab ea vigesima nona Libri primi Euclidaei (pauculis quibusdam exceptis) universa innititur Geometria. Alii vero (non sine magno in rigidam Logicam peccato) eas tales rectas lineas parallelas, nimirum *aequidistantes*, assumunt tanquam datas, ut inde gradum faciant ad reliqua demonstranda.

Preface to the Reader

Of all who have learned mathematics,[1] none can fail to know how great is the excellence and worth of Euclid's *Elements.* As erudite witnesses here I summon Archimedes, Apollonius, Theodosius, and others almost innumerable, writers on mathematics even to our times, who use Euclid's *Elements* as foundation long established and wholly unshaken. But this so great celebrity has not prevented many, ancients as well as moderns, and among them distinguished geometers, maintaining they had found certain blemishes[2] in these most beauteous nor ever sufficiently praised *Elements.* Three such flecks they designate, which now I name.

The first pertains to the definition of parallels and with it the Axiom which in Clavius[3] is the thirteenth of the First Book of the *Elements*, where Euclid says: *If a straight line falling on two straight lines, lying in the same plane, make with them two internal angles toward the same parts less than two right angles, these two straight lines infinitely produced toward those parts will meet each other*. No one doubts the truth of this Assertion;[4] but solely they accuse Euclid as to it, because he has used for it the name Axiom, as if obviously from the right understanding of its terms alone came conviction.[5] Whence not a few (withal retaining Euclid's definition of parallels) have attempted its demonstration from those propositions of Euclid's First Book alone which precede the 29th, wherein begins the use of the controverted Assertion.[6]

But again, since the endeavors of the ancients in this matter do not seem to attain the goal, so it has happened that many distinguished geometers of ensuing times, attacking the same idea, have thought necessary a new definition of parallels. Thus, while Euclid defines parallels' as straight lines *lying in the same plane, which, if infinitely produced toward both sides, nowhere meet,* they substitute for the last words of the given definition these others: *always equidistant from each* other;[7] so that all perpendiculars from any points on one of them let fall upon the other are equal to one another.

But again here arises a new fissure. For some, and these surely the keenest, endeavor to demonstrate the existence of parallel straight lines as so defined, whence they go up to the proof of the debated Assertion as stated in Euclid's terms, upon which truly from that *Elements* I, 29 (with some very few exceptions) all geometry rests. But others (not without gross sin against rigorous logic) assume such parallel straight lines, forsooth *equidistant,* as if given, that thence they may go up to what remains to be proved.

Et haec quidem satis sunt ad praemonendum Lectorem super iis, quae materiam exhibebunt Libro priori hujus mei Opusculi: Nam uberior praedictorum omnium explicatio habebitur in Scholiis post Prop. vigesimam primam enunciati Libri, quem dividam in duas veluti partes. In priore imitabor antiquiores illos Geometras, nihil propterea sollicitus de natura, aut nomine illius lineae, quae omnibus suis punctis a quadam supposita recta linea aequidistet: Sed unice in id incumbam, ut controversum Euclidaeum Axioma citra omnem petitionem principii clare demonstrem; nunquam idcirco adhibens ex ipsis prioribus Libri primi Euclidaei Propositionibus, non modo vigesimam septimam, aut vigesimam octavam, sed nec ipsas quidem decimam sextam, aut decimam septimam, nisi ubi clare agatur de tri-
[XI] angulo omni ex parte circumscripto. Tum in posteriore parte, ad novam ejusdem Axiomatis confirmationem demonstrabo non nisi rectam lineam esse posse, quae omnibus suis punctis a quadam supposita recta linea aequidistet. Horum autem occasione prima ipsa universae Geometriae Principia rigido examini subjicienda hic esse nullus est, qui non videat.

Transeo ad alios duos naevos Euclidi objectos. Prior respicit definitionem sextam Libri quinti super aeque proportionalibus: Posterior Definitionem quintam Libri sexti super compositione rationum. Hic autem erit secundi mei Libri unicus scopus, ut dilucide explicem praefatas Euclidaeas Definitiones, simulque ostendam non aequo jure hac in parte Euclidis nomen vexatum fuisse.

Prodest tamen rursum praemonere, demonstratum a me iri hac occasione unum quoddam Axioma, quod tutissime per omnem Geometriam versetur, sine indigentia illius *Postulati*, sub nomine Axiomatis ab interpretibus (ut reor) intrusi, cujus usus incipit ad 18. quinti.

And this is enough to indicate to the reader what will be the material of the First Book of this booklet of mine: for a more complete explication of all that has been said will be given in the Scholia after Proposition 21 of this First Book. I divide this Book into two parts. In the First Part I will imitate the antique geometers, and not trouble myself about the nature or the name of that line which at all its points is equidistant from a given straight line; but merely undertake without any *petitio principii* clearly to demonstrate the disputed Euclidean Axiom. Therefore never will I use from those prior propositions of *Elements* I, 27 and 28, but not even *Elements* I, 16 and 17, except where clearly it is question of a triangle every way restricted.[8] Then in the Second Part for a new confirmation of the same Axiom, I shall demonstrate that the line which at all its points is equidistant from a given straight line can only be a straight line. But every one sees that on this occasion the very first principles of all geometry are to be subjected to a rigid examination.

I go on to the other two blemishes charged against Euclid. The first pertains to Definition 6 of the Fifth Book of the *Elements* about equiproportionals; the second to Definition 5 of the Sixth Book about the composition of ratios.[9] It will be the sole aim of my Second Book to clearly expound the Euclidean definitions mentioned, and at the same time to show that Euclid's fame is here unjustly attacked.

Yet again it is well to state that on this occasion I shall prove a certain Axiom that may safely be applied throughout the whole of geometry, without need of that *Postulate,* put in (as I believe) by commentators under the name of Axiom,[10] whose use begins at *Elements* V, 18.

Indicis loco addenda censeo, quae sequuntur

In place of an Index should be added, I think, what follows

1. In Propositions 1 and 2 of the First Book two principles are established, from which in Propositions 3 and 4 is proved, that interior angles at the straight joining the extremities of equal perpendiculars erected toward the same parts (in the same plane) from two points of another straight, as base, not merely are equal to each other, but besides are either right or obtuse or acute according as that join is equal to, or less, or greater than the aforesaid base: and inversely. *From page 1 on.*
2. Hence occasion is taken to distinguish three different hypotheses, one of right angle, another of obtuse, a third of acute: about which in Propositions 5, 6, and 7 is proved, that any one of these hypotheses is always alone true if it is found true in any one particular case. *From page 5 on.*
3. Then after the interposition of three other necessary Propositions, is proved in Propositions 11, 12, and 13, the universal truth of the famous Axiom, respect being had to the first two hypotheses, one of right angle, and the other of obtuse; and at length in Proposition 14 is shown the absolute falsity of the hypothesis of obtuse angle. And here begins a lengthy battle against the hypothesis of acute angle, which alone opposes the truth of that Axiom. *From page 10 on.*
4. And so in Propositions 15 and 16 is proved that the hypothesis either of right angle, or obtuse, or acute is established from any rectilineal triangle whose three angles together are equal to, or greater, or less than two right angles; and in like way from any rectilineal quadrilateral, whose four angles are together equal to, or greater, or less than four right angles. *From page 20 on.*
5. Five other Propositions follow, in which are proved other indications for distinguishing the true hypothesis from the false. *From page 23 on.*
6. Now come four fundamental Scholia. In the last is exhibited a certain geometric diagram, of which Euclid perhaps thought, in order that his Assertion might assume self-evidence. In the preceding three is shown that the prior endeavors of distinguished geometers have not reached their aim. Since however the debated Axiom can be exactly proved from two straight lines presupposed *equidistant,* the author here shows a manifest *petitio principii* to be contained in that presupposition. If one wishes here to appeal to common persuasion, and surest *experience,* again he shows appeal should not be taken to an experience involving an infinity of points, when a single experiment pertaining to any one point can suffice. In this place are set forth by him three invincible physico-geometric demonstrations. *From page 29 on.*
7. To the end of the First Part of this Book there remain twelve other Propositions. I do not state the particular assumptions, because they are too complex. I only say here at length I have disproved the hostile hypothesis of acute angle by a manifest falsity, since it must lead to the recognition of two straight lines which at one and the same point have in the same plane a common perpendicular. That this is contrary to the nature of the straight line is proved by five Lemmata, in which are contained five fundamental Axioms relating to the straight line and circle, with their correlative postulates. *From page 43 on.*

8. Secunda pars continet sex Propositiones. Ibi autem; post expensam (juxta hypothesim anguli acuti) naturam illius lineae, quae omnibus suis punctis a quadam praesupposita recta linea aequidistet; multis modis ostenditur, eam fore aequalem contrapositae basi, unde infertur praenunciatae hypothesis certissima falsitas. Quare tandem in ultima Propos. quae est XXXIX. exactissime demonstratur celebre illud Euclidaeum Axioma, cui nempe (ut omnes sciunt) universa Geometria innititur. *a pag. 87*
9. Secundus Liber digeri commode non potuit in Propositiones, etiamsi locis opportunis plura intermista sint utilissima Theoremata, ac Problemata. Meretur nihilominus expresse notari unum quoddam Axioma, cujus ibi demonstratur non modo veritas, [XV] verum etiam universalis utilitas pro omni Geometria, sine indigentia alterius parum decori Postulati, quod ab interpretibus censeri potest intrusum sub nomine Axiomatis, cujus nempe usus incipit ad 18. quinti. Et id quidem pro prima Parte hujus Libri, in qua vindicatur Def. sexta quinti Euclidaei. *a pag. 102*
10. Tum in secunda Parte; praeter nonnulla alia opportune addita, ad tuendas reliquas Definitiones ejusdem Quinti circa magnitudines proportionales; demonstratur priore loco (respectu habito ad magnitudines commensurabiles) quinta Definitio Sexti, etiamsi recipi ea deberet in *quid rei*, veluti Axioma: Sed rursum multis exemplis, ex ipso Euclide petitis, ostenditur nullius demonstrationis indigam eam esse, quia Definitionem *puri nominis*. Atque ita, post opportunam additam Appendicem, huic Operi finis imponitur. *a pag. 132*

8. The Second Part contains six Propositions. Here, after investigating the nature (assuming the hypothesis of acute angle) of that line which at all its points is equidistant from a given straight line, it is shown in many ways that it equals the base opposite, whence is inferred the certain falsity of the aforesaid hypothesis. Wherefore at length in the last Proposition 39 is exactly proved that famous Axiom of Euclid, upon which (as everybody knows) the whole of geometry rests. *From page* 87 *on.*
9. The Second Book cannot conveniently be divided into Propositions, although at opportune places are intercalated many most useful theorems and problems. Nevertheless is worthy of express mention a certain Axiom, of which not merely the truth is there demonstrated but also the universal utility for all geometry, without need of the other inelegant postulate supposedly inserted by commentators under the name of Axiom, whose use begins at *Elements* V, 18. So much for the First Part of this Book, in which is defended *Elements* V, Def. 6. *From page 102 on*.
10. Then in the Second Part, besides some other things opportunely added for the purpose of maintaining other Definitions of *Elements* V about proportional magnitudes, is demonstrated in the first place (with respect to commensurable magnitudes) *Elements* VI, Def. 5, even if it ought to be taken as a *real* definition, i. e. as an axiom. But on the contrary is shown by many examples drawn from Euclid himself that this needs no demonstration, because it is a *purely nominal* definition. And so after an Appendix opportunely added, an end is put to this work. *From page 132 on*.