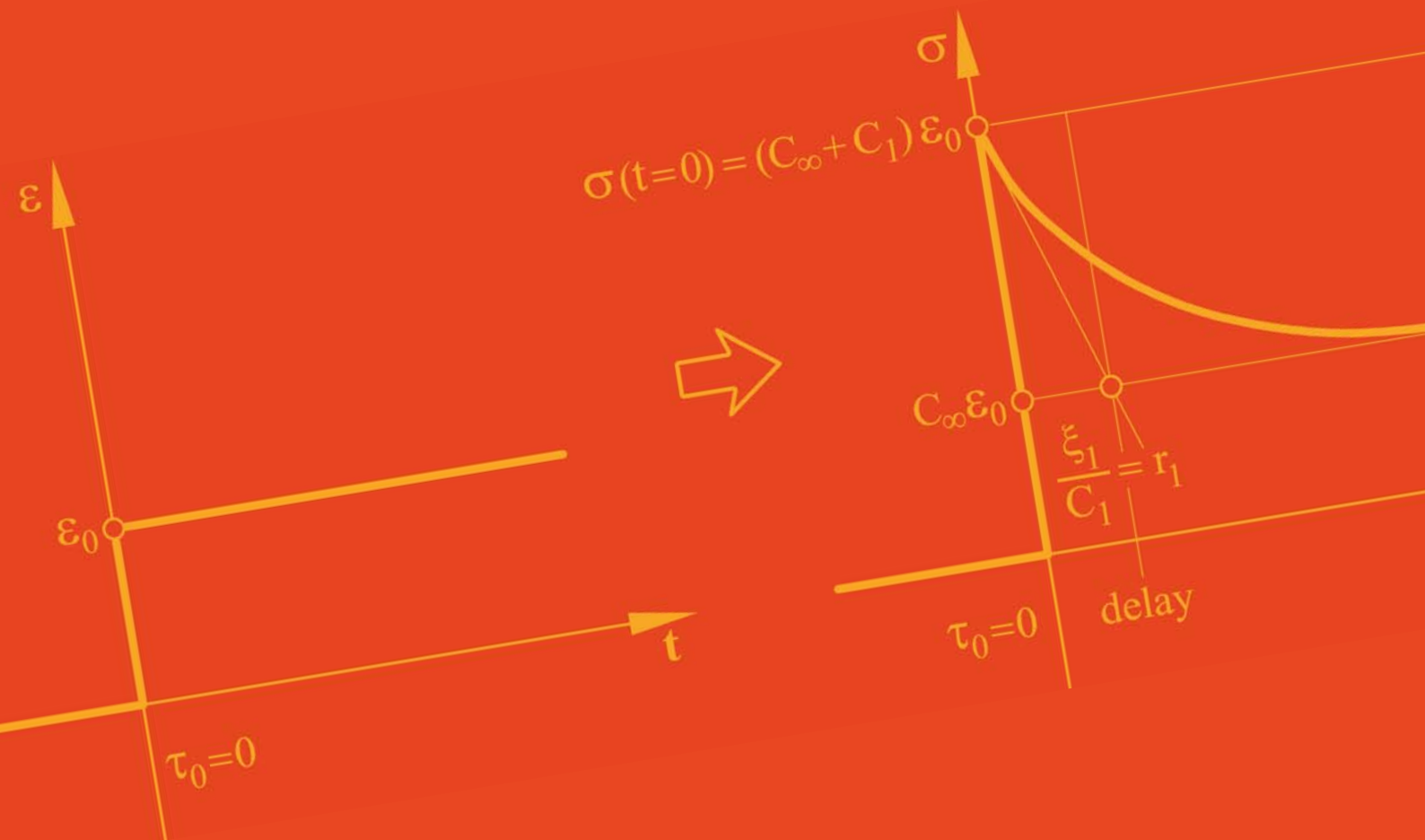


# Nonlinear Dynamics of Structures

Sergio Oller



# **Nonlinear Dynamics of Structures**

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# Lecture Notes on Numerical Methods in Engineering and Sciences

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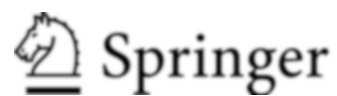
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Sergio Oller

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*This work is dedicated to my wife and son, and also to all my loved ones*



# Preface

This book has been written to present the conceptual basis of “Nonlinear Dynamics” of structural systems. Although there are many papers on this subject, I have decided to write this book for educational purposes addressed to students with an academic level equivalent to a master’s degree.

The book is divided into three main parts: the first one sets up the basis on which the nonlinear dynamics applied to discrete structures is based on; the second one shows the effect of time-independent constitutive model behavior within the nonlinear dynamic response; and finally, the third part analyzes the effect of time-dependent constitutive models in a nonlinear dynamic behavior.

This work has been possible thanks to the institutional support of CIMNE (International Center for Numerical Methods in Engineering), which has financially supported this book since its first edition in Spanish in 2002, and later in its English edition. Many people have participated in the latter, and I would particularly like to thank Ms. Hamdy Briceño, Prof. Miguel Cerrolaza and Cristina Pérez Arias for their careful translation and revision of this text. I would also like to thank all my students who have contributed to the correction of the text during the eleven years that this book has been used as a syllabus of the “Nonlinear Dynamics” course in the Department of Strength of Materials, at the Technical University of Catalonia, Spain.

I hope these notes will contribute to a better understanding of the nonlinear dynamics and encourage the reader to study this subject in greater depth.

Barcelona, May 2014

*Sergio Oller*





# Contents

<b>1 Introduction .....</b>	<b>1-1</b>
<b>2 Thermodynamic Basis of the Equation of Motion .....</b>	<b>2-1</b>
2.1 Introduction.....	2-1
2.2 Kinematics of deformable bodies.....	2-1
2.2.1 Basic definitions of tensors describing the kinematics of a point in the space .....	2-1
2.2.2 Strain measurements.....	2-3
2.2.3 Relationships among mechanical variables.....	2-4
2.2.4 The objective derivative.....	2-6
2.2.5 Velocity.....	2-6
2.2.6 Stress measurements.....	2-7
2.3 Thermodynamics basics.....	2-8
2.3.1 First law of thermodynamics.....	2-8
2.3.2 Second law of thermodynamics.....	2-11
2.3.3 Lagrangian local form of mechanical dissipation.....	2-13
2.4 Internal variables.....	2-15
2.5 Dynamic equilibrium equation for a discrete solid.....	2-15
2.5.1 Nonlinear problem – Linearization of the equilibrium equation .....	2-18
2.6 Different types of nonlinear dynamic problems.....	2-20
2.6.1 Material nonlinearities.....	2-23
<b>3 Solution of the Equation of Motion.....</b>	<b>3-1</b>
3.1 Introduction.....	3-1
3.2 Explicit-implicit solution.....	3-3
3.3 Implicit solution.....	3-4
3.3.1 Equilibrium at time $(t + \Delta t)$ .....	3-4
3.3.2 Equilibrium solution in time – Implicit methods.....	3-5
3.3.2.1 The Newmark method.....	3-6
3.3.2.2 The Houbolt method.....	3-11
3.3.3 Solution of the nonlinear-equilibrium equations system.....	3-12
3.3.3.1 The Newton-Raphson method.....	3-13
3.3.3.2 The modified Newton-Raphson method.....	3-14
3.3.3.3 Convergence accelerators.....	3-15
3.3.3.4 Aitken’s accelerator or extrapolation algorithm.....	3-15
3.3.3.5 B.F.G.S. algorithms .....	3-16
3.3.3.6 The Secant-Newton algorithms.....	3-17
3.3.3.7 “Line-Search” algorithms.....	3-18
3.3.3.8 Solution control algorithms – “Arc-Length”.....	3-20

<b>4 Convergence Analysis of the Dynamic Solution. ....</b>	<b>4-1</b>
4.1 Introduction.....	4-1
4.2 Reduction to the linear elastic problem.....	4-1
4.3 Solution of second-order linear symmetric systems.....	4-4
4.4 The dynamic equilibrium equation and its convergence-consistency and stability... ..	4-5
4.5 Solution stability of second-order linear symmetric systems.....	4-6
4.5.1 Stability analysis procedure.....	4-6
4.5.2 Determination of <b>A</b> and <b>L</b> for Newmark.....	4-7
4.5.3 Determination of <b>A</b> and <b>L</b> for central differences- Newmark's explicit form .....	4-11
4.6 Solution stability of second-order nonlinear symmetric systems.....	4-13
4.6.1 Stability of the linearized equation .....	4-13
4.6.2 Energy conservation algorithms.....	4-14
APPENDIX - 1 .....	4-18
APPENDIX - 2 .....	4-24

<b>5 Time-independent Models. ....</b>	<b>5-1</b>
5.1 Introduction.....	5-1
5.2 Elastic behavior.....	5-1
5.2.1 Invariants of the tensors.....	5-4
5.3 Nonlinear elasticity.....	5-5
5.3.1 Introduction.....	5-5
5.3.2 Nonlinear hyper-elastic model.....	5-6
5.3.2.1 Stress-based hyper-elastic model.....	5-6
5.3.2.2 Stability postulates.....	5-7
5.4 Plasticity in small deformations.....	5-10
5.4.1 Introduction.....	5-10
5.4.2 Discontinuity behavior or plastic yield criterion .....	5-12
5.5 Elasto-plastic behavior.....	5-15
5.5.1 The Levy-Mises theory .....	5-15
5.5.2 The Prandtl-Reus theory.....	5-16
5.6 The classic plasticity theory.....	5-17
5.6.1 Plastic unit or specific work.....	5-17
5.6.2 Plastic loading surface. Plastic hardening variable.....	5-19
5.6.2.1 Isotropic hardening.....	5-20
5.6.2.2 Kinematic hardening.....	5-21
5.6.3 Stress-strain relation. Plastic consistency and tangent stiffness.....	5-22
5.7 Drucker's stability postulate and maximum plastic dissipation.....	5-23
5.8 Stability condition.....	5-25
5.8.1 Local stability.....	5-25
5.8.2 Global stability.....	5-26
5.9 Condition of unicity of solution.....	5-27
5.10 Kuhn-Tucker. Loading-unloading condition.....	5-27
5.11 Yield or plastic discontinuity classic criteria.....	5-28
5.11.1 The Rankine criterion of maximum tension stress.....	5-29
5.11.2 The Tresca criterion of maximum shear stress.....	5-31
5.11.3 The Von Mises criterion of octahedral shear stress.....	5-33
5.11.4 The Mohr-Coulomb criterion of octahedral shear stress.....	5-35
5.11.5 The Drucker-Prager criterion .....	5-39

5.12 Geomaterials' plasticity. ....	5-41
5.13 Basis of the plastic-damage model. ....	5-42
5.13.1 Mechanical behavior required for the constitutive model formulation....	5-44
5.13.2 Some characteristics of the plastic damage model.....	5-44
5.14 Main variables of the plastic-damage model.....	5-47
5.14.1 Definition of the plastic damage variable.....	5-47
5.14.2 Definition of the law of evolution of cohesion $c - \kappa^p$ . ....	5-50
5.14.3 Definition of the variable $\phi$ , internal friction angle.....	5-51
5.14.4 Variable definition $\psi$ , dilatancy angle. ....	5-55
5.15 Generalization of the damage model with stiffness degradation.....	5-56
5.15.1 Introduction.....	5-56
5.15.2 Elasto-plastic constitutive equation with stiffness degradation. ....	5-58
5.15.3 Tangent constitutive equation for stiffness degradation processes .....	5-60
5.15.4 Particular yield functions. ....	5-61
5.15.4.1 The Mohr-Coulomb modified function.....	5-61
5.15.4.2 The Drucker-Prager modified function. ....	5-64
5.16 Isotropic continuous damage - Introduction.....	5-66
5.16.1 Isotropic damage model.....	5-67
5.17 Helmholtz's free energy and constitutive equation.....	5-68
5.18 Damage threshold criterion.....	5-69
5.19 Evolution law of the internal damage variable. ....	5-70
5.20 Tangent constitutive tensor of damage. ....	5-71
5.21 Particularization of the damage criterion. ....	5-72
5.21.1 General softening. ....	5-72
5.21.2 Exponential softening. ....	5-73
5.21.3 Linear softening. ....	5-74
5.22 Particularization of the stress threshold function. ....	5-75
5.22.1 The Simo -Ju model.....	5-75
5.22.1.1 Setting of the $A$ parameter for the Simo-Ju model.....	5-75
5.22.2 The Lemaitre and Mazars model.....	5-77
5.22.3 General model for different damage surfaces. ....	5-78
5.22.4 Setting of the $A$ parameter.....	5-78

## **6 Time-dependent Models.....6-1**

6.1 Introduction.....	6-1
6.2 Constitutive equations based on spring-damping analogies. ....	6-1
6.2.1 The simplified Kelvin model .....	6-2
6.2.2 The simplified Maxwell model.....	6-3
6.2.3 The generalized Kelvin model.....	6-4
6.2.4 The generalized multiple Kelvin model.....	6-5
6.2.5 The generalized Maxwell model .....	6-7
6.2.6 The generalized multiple Maxwell model.....	6-9
6.2.6.1 Dissipation evaluation.....	6-11
6.3 Multiaxial generalization of the viscoelastic constitutive laws.....	6-12
6.3.1 Multiaxial form of viscoelastic models. ....	6-12
6.3.2 Numerical solution of integrals and algorithms. ....	6-13
6.4 The Kelvin model in dynamic problems. ....	6-16
6.4.1 The Kelvin model dissipation. ....	6-17
6.4.2 Equation of the dynamic equilibrium for the Kelvin model.....	6-17
6.4.3 Stress considerations. The Rayleigh vs. the Kelvin model. ....	6-19
6.4.4 Dissipation considerations. The Rayleigh vs. the Kelvin model.....	6-19

6.4.5	Cantilever beam .....	6-21
6.4.6	Frame with rigid beam and lumped mass. ....	6-22
6.5	Viscoplasticity.....	6-24
6.5.1	Limit states of viscoplasticity. ....	6-26
6.5.2	Over stress function. ....	6-27
6.5.3	Integration algorithm for the viscoplastic constitutive equation. ....	6-27
6.5.4	The particular case of the Duvaut-Lyon model for a Von Mises viscoplastic material .....	6-28