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Fumio Hiai · Dénes Petz

Introduction to Matrix Analysis and Applications

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Preface

The material of this book is partly based on the lectures of the authors given at the Graduate School of Information Sciences of Tohoku University and at the Budapest University of Technology and Economics. The aim of the lectures was to explain certain important topics in matrix analysis from the point of view of functional analysis. The concept of Hilbert space appears many times, but only finite-dimensional spaces are used. The book treats some aspects of analysis related to matrices including such topics as matrix monotone functions, matrix means, majorization, entropies, quantum Markov triplets, and so on. There are several popular matrix applications in quantum theory.

The book is organized into seven chapters. [Chapters 1–3](#) form an introductory part of the book and could be used as a textbook for an advanced undergraduate special topics course. The word “matrix” was first introduced in 1848 and applications subsequently appeared in many different areas. [Chapters 4–7](#) contain a number of more advanced and less well-known topics. This material could be used for an advanced specialized graduate-level course aimed at students who wish to specialize in quantum information. But the best use for this part is as a reference for active researchers in the field of quantum information theory. Researchers in statistics, engineering, and economics may also find this book useful.

[Chapter 1](#) introduces the basic notions of matrix analysis. We prefer the Hilbert space concepts, so complex numbers are used. The spectrum and eigenvalues are important, and the determinant and trace are used later in several applications. The final section covers tensor products and their symmetric and antisymmetric subspaces. The chapter concludes with a selection of exercises. We point out that in this book “positive” means ≥ 0 ; we shall not use the term “non-negative.”

[Chapter 2](#) covers block matrices, partial ordering, and an elementary theory of von Neumann algebras in the finite-dimensional setting. The Hilbert space concept requires projections, i.e., matrices P satisfying $P = P^2 = P^*$. Self-adjoint matrices are linear combinations of projections. Not only are single matrices required, but subalgebras of matrices are also used. This material includes Kadison’s inequality and completely positive mappings.

[Chapter 3](#) details matrix functional calculus. Functional calculus provides a new matrix $f(A)$ when a matrix A and a function f are given. This is an essential tool in matrix theory as well as in operator theory. A typical example is the exponential

function $e^A = \sum_{n=0}^{\infty} A^n/n!$ If f is sufficiently smooth, then $f(A)$ is also smooth and we have a useful Fréchet differential formula.

Chapter 4 covers matrix monotone functions. A real function defined on an interval is matrix monotone if $A \leq B$ implies $f(A) \leq f(B)$ for Hermitian matrices A and B whose eigenvalues are in the domain of f . We have a beautiful theory of such functions, initiated by Löwner in 1934. A highlight is the integral expression of such functions. Matrix convex functions are also considered. Graduate students in mathematics and in information theory will benefit from having all of this material collected into a single source.

Chapter 5 covers matrix (operator) means for positive matrices. Matrix extensions of the arithmetic mean $(a + b)/2$ and the harmonic mean

$$\left(\frac{a^{-1} + b^{-1}}{2}\right)^{-1}$$

are rather trivial, however it is nontrivial to define matrix version of the geometric mean \sqrt{ab} . This was first done by Pusz and Woronowicz. A general theory of matrix means developed by Kubo and Ando is closely related to operator monotone functions on $(0, \infty)$. There are also more complicated means. The mean transformation $M(A, B) := m(\mathbb{L}_A, \mathbb{R}_B)$ is a mean of the left-multiplication \mathbb{L}_A and the right-multiplication \mathbb{R}_B , recently studied by Hiai and Kosaki. Another useful concept is a multivariable extension of two-variable matrix means.

Chapter 6 discusses majorizations for eigenvalues and singular values of matrices. Majorization is a certain order relation between two real vectors. **Section 6.1** recalls classical material that can be found in other sources. There are several famous majorizations for matrices which have strong applications to matrix norm inequalities in symmetric norms. For instance, an extremely useful inequality is the Lidskii–Wielandt theorem.

The last chapter contains topics related to quantum applications. Positive matrices with trace 1, also called density matrices, are the states in quantum theory. The relative entropy appeared in 1962, and matrix theory has found many applications in the quantum formalism. The unknown quantum states can be described via the use of positive operators $F(x)$ with $\sum_x F(x) = I$. This is called a POVM and a few mathematical results are shown, but in quantum theory there are much more relevant subjects. These subjects are close to the authors' interests, and there are some very recent results.

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April 2013

Fumio Hiai
Dénes Petz

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