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Superconcentration and Related Topics

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Preface

Understanding the fluctuations of random objects is one of the major goals of probability theory. There is a whole subfield of probability and analysis, called concentration of measure, devoted to understanding fluctuations of random objects. Measure concentration has seen tremendous progress in the last forty years. And yet, there is a large class of problems in which classical concentration of measure gives suboptimal bounds on the order of fluctuations.

In 2008 and 2009, I posted two preprints on arXiv where it was shown that the suboptimality of classical concentration, when it occurs, is not simply a question of mathematical inadequacy. The suboptimality is in fact equivalent to a number of very interesting things going on in the structure of the random object under investigation. Indeed, the consequences are possibly interesting enough for the suboptimality of classical concentration to deserve a name of its own; I call it 'superconcentration'.

This monograph is a combination of these two preprints (which will not be published individually), together with some new material and new insights. The majority of the results are the same as in the preprints, but the presentation is radically different. In particular, I think I achieved a substantial degree of simplification and clarity through the use of the spectral approach. This is quite standard in the noise-sensitivity literature (which is intimately connected with the topic of this monograph), but it is not the way I derived the results in the preprints.

In addition to the theorems and proofs, I have interspersed the document with a sizable number of open problems for professional mathematicians and exercises for graduate students.

I spent many hours deliberating over whether to keep the book in its present form, or expand it to around 250 pages by including additional material from classical concentration of measure and other related topics. In the end I decided not to expand. The rationale behind this decision is two-fold: The first reason is that there are several comprehensive texts on concentration of measure already available in the market, and I did not wish to encroach on that territory. I had originally intended this book to be a short and succinct exposition of the superconcentration phenomenon, and in the end I decided to keep it that way. The second reason is that I am deeply familiar with my procrastinating tendencies, which made me confident that I would never have finished this project if I had planned a major overhaul. However I did expand a little bit; the original version that I submitted to Springer was even thinner. On the advice of one of the reviewers, I decided to include several additional examples that I had omitted in the first draft.

The main body of this monograph grew out of a set of six lectures I gave at the Cornell Probability Summer School in July 2012. The task of explaining the results and proofs to graduate students forced me to organize the material in a manner suitable for exposition in a monograph. I thank the organizers of CPSS 2012 for giving me this opportunity, and one of the students attending the summer school, Mihai Nica, for doing a terrific job in taking notes and typing them up.

I am grateful to Persi Diaconis for suggesting that I write this monograph, and for his constant encouragement and advice. I thank the reviewers for many useful comments, and Christophe Garban, Susan Holmes, Dmitry Panchenko and Michel Talagrand for looking at the early drafts, giving suggestions for improvements, and pointing out misattributions and errors. Dr. Catriona Byrne of Springer and her editorial team has my gratitude for being exceptionally helpful and responsive in every step of the preparation of the manuscript.

I would like to acknowledge the role played by the National Science Foundation, the Courant Institute, and UC Berkeley in funding, at various stages, the research related to this book.

And finally, I must thank one person who is outside the realm of mathematics and yet played an indispensable role in the completion of this project: I would have never finished writing the monograph without the sustained urging, care and patience of my wife, Esha. I thank her for all that and much more.

Stanford, CA, USA Sourav Chatterjee

Contents

1 Introduction . . . 1
1 Superconcentration . . . 1
2 Chaos . . . 7
3 Multiple Valleys . . . 11

2 Markov Semigroups . . . 15
1 Semigroup Basics . . . 15
2 The Ornstein-Uhlenbeck Semigroup . . . 17
3 Connection with Malliavin Calculus . . . 18
4 Poincaré Inequalities . . . 18
5 Some Applications of the Gaussian Poincaré Inequality . . . 19
6 Fourier Expansion . . . 20

3 Superconcentration and Chaos . . . 23
1 Definition of Superconcentration . . . 23
2 Superconcentration and Noise-Sensitivity . . . 25
3 Definition of Chaos . . . 25
4 Equivalence of Superconcentration and Chaos . . . 27
5 Some Applications of the Equivalence Theorem . . . 28
6 Chaotic Nature of the First Eigenvector . . . 29

4 Multiple Valleys . . . 33
1 Chaos Implies Multiple Valleys: The General Idea . . . 33
2 Multiple Valleys in Gaussian Polymers . . . 34
3 Multiple Valleys in the SK Model . . . 36
4 Multiple Peaks in Gaussian Fields . . . 38
5 Multiple Peaks in the *NK* Fitness Landscape . . . 40

5 Talagrand's Method for Proving Superconcentration . . . 45
1 Hypercontractivity . . . 45
2 Talagrand's L^1–L^2 Bound . . . 47
3 Talagrand's Method Always Works for Monotone Functions . . . 49

4 The Benjamini-Kalai-Schramm Trick 51
5 Superconcentration in Gaussian Polymers 54
6 Sharpness of the Logarithmic Improvement 56

6 The Spectral Method for Proving Superconcentration 57
1 Spectral Decomposition of the OU Semigroup 58
2 An Improved Poincaré Inequality 60
3 Superconcentration in the SK Model 60

7 Independent Flips 63
1 The Independent Flips Semigroup 63
2 Hypercontractivity for Independent Flips 65
3 Chaos Under Independent Flips 66

8 Extremal Fields 73
1 Superconcentration in Extremal Fields 73
2 A Sufficient Condition for Extremality 78
3 Application to Spin Glasses 79
4 Application to the Discrete Gaussian Free Field 83

9 Further Applications of Hypercontractivity 87
1 Superconcentration of the Largest Eigenvalue 87
2 A Different Hypercontractive Tool 90
3 Superconcentration in Low Correlation Fields 92
4 Superconcentration in Subfields 93
5 Discrete Gaussian Free Field on a Torus 94
6 Gaussian Fields on Euclidean Spaces 99

10 The Interpolation Method for Proving Chaos 105
1 A General Theorem 105
2 Application to the Sherrington-Kirkpatrick Model 111
3 Sharpness of the Interpolation Method 112

11 Variance Lower Bounds 115
1 Some General Tools 115
2 Application to the Edwards-Anderson Model 118
3 Chaos in the Edwards-Anderson Model 119

12 Dimensions of Level Sets 125
1 Level Sets of Extremal Fields 125
2 Induced Dimension 128
3 Dimension of Near-Maximal Sets 131
4 Applications 134

Appendix A Gaussian Random Variables 137
1 Tail Bounds 137
2 Size of the Maximum 137
3 Integration by Parts 139
4 The Gaussian Concentration Inequality 139
5 Concentration of the Maximum 140

Appendix B Hypercontractivity 143

References 147

Author Index 153

Subject Index 155