Compact Textbooks in Mathematics

For further volumes:
http://www.springer.com/series/11225

Birkhäuser

Compact Textbooks in Mathematics

This textbook series presents concise introductions to current topics in mathematics and mainly addresses advanced undergraduates and master students. The concept is to offer small books covering subject matter equivalent to 2- or 3-hour lectures or seminars which are also suitable for self-study. The books provide students and teachers with new perspectives and novel approaches. They feature examples and exercises to illustrate key concepts and applications of the theoretical contents. The series also includes textbooks specifically speaking to the needs of students from other disciplines such as physics, computer science, engineering, life sciences, finance.

Gunther Leobacher
Friedrich Pillichshammer

Introduction to Quasi-Monte Carlo Integration and Applications

Birkhäuser

Gunther Leobacher
Institute of Financial Mathematics
University of Linz
Linz, Austria

Friedrich Pillichshammer
University of Linz
Linz, Austria

ISBN 978-3-319-03424-9 ISBN 978-3-319-03425-6 (eBook)
DOI 10.1007/978-3-319-03425-6
Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014945104

2010 Mathematics Subject Classification : 11K06, 11K38, 11K45, 65C05, 65D30

Cover design: deblik, Berlin

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

To Charlotte and Gisi

Preface

While Fubini's theorem states that the integral of a function on the s-dimensional unit cube can be computed simply by computing iterated integrals, the attempt of doing so for a function for which those integrals cannot be given by closed-form formulas is in most cases doomed to fail, if s, say, is greater than 10. The reason for this is that by iteration of a one-dimensional integration rule the number of function evaluation needed for the corresponding product rule grows exponentially in s.

This constraint on the practical computation of integrals led to the development of probabilistic methods. Here, the integral is interpreted as the expected value of the integrand evaluated at a random variable that is uniformly distributed on the s-dimensional unit cube. These methods were first applied by E. Fermi, S. Ulam and J. von Neumann, the latter also being the originator of the name "Monte Carlo simulation".

In contrast to Monte Carlo integration rules, which sample the integrand at random points, so-called *quasi*-Monte Carlo rules use deterministic sample points. The relationship between Monte Carlo and quasi-Monte Carlo corresponds to the relationship between two notions of "uniform distribution" in mathematics. The first is the probabilistic notion of a random variable for which the probability of taking values in a given subset of the unit cube is precisely the volume of that set. The second notion is that of a sequence of points, for which the proportion of points of the sequence lying in a given s-dimensional sub-interval of the unit cube equals the volume of the sub-interval.

While the notion of uniformly distributed sequences and examples thereof had been coined earlier, the birth of the theory of "Uniform Distribution Modulo One" is marked by H. Weyl's seminal paper "Über die Gleichverteilung mod. Eins", first published in the year 1916. It was already known by then that, at least in principle, uniformly distributed sequences could be used to integrate Riemann-integrable functions. However, under the weak assumptions the convergence of the sample mean to the integral can be arbitrarily slow, making the "method" impractical.

As the starting point for the analysis of quasi-Monte Carlo methods for numerical integration one can consider the establishment of the Koksma-Hlawka inequality, which was shown by J.F. Koksma in 1942 for the one-dimensional case and by E. Hlawka in 1961 for arbitrary dimensions. Since then the Koksma-Hlawka inequality is the prototypical error estimate for quasi-Monte Carlo integration. Its main feature is that it bounds the integration error by the product of two terms, the variation of the function and the star discrepancy of the underlying sample nodes. The second notion is related to that of uniform distribution of a sequence, but while the latter is an asymptotic quality, the star discrepancy allows to assess the quality of uniformity of a finite number of points. Knowing how well the points can be chosen with respect to that measure means – thanks to the Koksma-Hlawka inequality – knowing the possible convergence of the integration error. This is where concepts from Discrepancy Theory enter the game.

From the early 1960s on several people, among these N.M. Korobov, E. Hlawka, I.M. Sobol', J. Halton, H. Faure, H. Niederreiter and C.P. Xing provided constructions of point sets and sequences with excellent distribution properties, i.e., with low star discrepancy or related/alternative quality measures. The point sets and sequences constructed in this way are therefore suitable sample points for quasi-Monte Carlo rules. However, a certain disadvantageous dependence of the discrepancy bounds on the dimension led to the belief that quasi-Monte Carlo rules can only be applied in very moderate dimensions. Contrary to these opinions, quasi-Monte Carlo rules are nowadays used for numerical integration of functions in hundreds or even thousands of dimensions, and since recently there is also a stream of research which studies infinite-dimensional integration. The motivation for this paradigm change lies in results of numerical experiments published in 1995 by S.H. Paskov and J.F. Traub, who studied quasi-Monte Carlo rules for functions in 360 dimensions coming from Mathematical Finance. But, despite their apparent effectivity even for those very high-dimensional problems, the question of exactly *why* quasi-Monte Carlo rules should give these good results is still not completely resolved. In 2010, at the MCQMC meeting in Warsaw, I.H. Sloan spoke in this context about "The unreasonable effectiveness of quasi-Monte Carlo". Although in the meantime some partial answers come from the study of weighted function spaces and from tractability theory, the quest for an explanation of this unreasonable effectiveness of quasi-Monte Carlo is still a very active part of research.

As suggested by its title, this book is an introductory text to quasi-Monte Carlo methods and some of their applications, and it aims at giving a comprehensible treatment of the subject with detailed explanations of the basic concepts. Originating from a 2-h one semester undergraduate course, it should be accessible to students in mathematics or computer science with basic knowledge of algebra, calculus, linear algebra, and probability theory. Although the main focus is on the theory behind the concepts of quasi-Monte Carlo, several practical applications with an emphasis on financial problems are discussed.

The topics of the book roughly retrace the history of quasi-Monte Carlo methods as sketched above, but do so using up-to-date concepts and notations. Thus we start with the classical multi-dimensional integration problem and its first high-dimensional alternative, Monte Carlo integration. Chapter 2 is devoted to uniform distribution of sequences and several concepts of discrepancy. We give a discrepancy estimate for one of the oldest specimens of low discrepancy sequences, the Halton sequence. In Chap. 3 we introduce the modern framework of reproducing kernel Hilbert spaces for obtaining bounds on the integration error for functions in those spaces. The Koksma-Hlawka inequality, though not in its most general form, appears as a special case of that theory. The next two chapters are mostly devoted to constructions of low-discrepancy point sets and sequences, namely lattice point sets, (t, m, s)-nets and (t, s)-sequences. The chapter on lattice rules includes a section on integration in weighted Korobov spaces. The concept of weighted spaces has some bearing on the issue of effectiveness of quasi-Monte Carlo methods for very high-dimensional problems. Chapter 6 concludes the theoretical part by providing more information about the curse of dimensionality and tractability of discrepancy.

The last two chapters constitute the application part of the book. Chapter 7 gives a very condensed introduction to concepts from Mathematical Finance, in particular derivative pricing. We introduce some models and derivatives that can serve as specimens for trying out the simulation methods provided in Chap. 8. This last chapter covers some of the basics of simulation, like generation of non-uniform random variables and generation of Brownian paths. The emphasis is on (fast-)orthogonal transforms for speeding up convergence. Several examples serve to illustrate the methods.

The compilation of a textbook demands a great deal not only from the authors, but also from their families, colleagues, and students, some of the common time of which has to be diverted to the project. We want to thank all of them for their support and understanding.

We appreciate valuable comments, suggestions and improvements from several colleagues which we would like to mention here: Josef Dick, Aicke Hinrichs, Peter Kritzer, Gerhard Larcher, Harald Niederreiter, Klaus Ritter and Wolfgang Ch. Schmid.

We hope that the book will turn out to be useful for teaching, self-study, and as a reference, and that it will encourage many people to study quasi-Monte Carlo methods and/or apply them to problems from Mathematical Finance or other areas.

Linz, Austria
October 2013

Gunther Leobacher
Friedrich Pillichshammer

Contents