For further volumes:
http://www.springer.com/series/304

Péter Major

Multiple Wiener-Itô Integrals

With Applications to Limit Theorems

Second Edition

Springer

Péter Major
Hungarian Academy of Sciences
Alfréd Rényi Mathematical Institute
Budapest, Hungary

ISBN 978-3-319-02641-1 ISBN 978-3-319-02642-8 (eBook)
DOI 10.1007/978-3-319-02642-8
Springer Cham Heidelberg New York Dordrecht London

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2013953865

Mathematics Subject Classification (2010): Primary: 60G18, 60H05, 60F99,
secondary: 60G10, 60G15, 60G60

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface to the Second Edition

This text is a slightly modified version of my Lecture Note *Multiple Wiener–Itô integrals with applications to limit theorems* published in the *Lecture Notes in Mathematics* series (number 849) of the Springer Verlag in 1981. I decided to write a revised version of this Lecture Note after a special course I held about its subject in the first semester of the academic year 2011–2012 at the University of Szeged. Preparing for this course I observed how difficult the reading of formulas in this Lecture Note was. These difficulties arose because this Lecture Note was written at the time when the TEX program still did not exist, and the highest technical level of typing was writing on an IBM machine that enabled one to type beside the usual text also mathematical formulas. But the texts written in such a way are very hard to read. To make my text more readable, I decided to retype it by means of the TEX program. But it turned out that a real improvement of the text demands much more than producing nice, readable formulas. To make a really better version of this work, I also had to explain better the results and definitions together with the ideas and motivation behind them. Besides, I had to make not only more readable formulas but also more readable explanations. The reader must see at each point of the discussion what is just going on and why. In the new version of this work, I tried to satisfy these demands. Naturally, I also corrected the errors I found. At some points I had to insert a rather long explanation in the proof, because I met such a statement which seemed to be trivial at the first sight, but its justification demanded a detailed discussion. I hope that these insertions did not make the work less transparent.

There appeared many new results about the subject of this Lecture Note since its first appearance. The question arose naturally whether I should insert them to the new edition of this work. Finally I decided to make no essential changes in the text, to restrict myself to the correction of the errors I found and to give a more detailed explanation of the proofs where I felt that it is useful. In making such a decision I

The preparation of this revised version was supported by the Hungarian National Foundation for Science (OTKA) under grant number 91928.

was influenced by a Russian proverb which says: “Лучше враг хорошего”. I tried to follow the advice of this proverb. (I do not know of an English counterpart of this proverb, but it has a French version: “Le mieux est l’ennemi du bien”.)

I made one exception. I decided to explain those basic notions and results in the theory of generalized functions which were applied in the older version of this work in an implicit way. In particular, I tried to explain with their help how one gets those results about the spectral representation of the covariance function of stationary random fields that I have presented under the names *Bochner’s theorem* and *Bochner–Schwartz theorem.* This extension of the text is contained in the attachments to Chaps. 1 and 3. In the first version I only referred to a work where these notions and results can be found. But now I found such an approach not satisfactory, because these notions and results play an important role in some arguments of this work. Hence I felt that to make a self-contained presentation of the subject, I have to explain them in more detail.

The first edition of this Lecture Note appeared long time ago, but the main question discussed in it, the description of the limit behaviour of appropriately normalized partial sums of strongly dependent random variables remained an open problem. Also the method applied in this work remained an important tool in the study of such problems. Hence a self-contained explanation of the theory which provides a good foundation for this method is useful. By my hopes this Lecture Note contains such an explanation, and therefore it did not become out of date. This was the main argument for myself to write a new version of this work where I tried to present a better and more accessible discussion.

I would like to write some words about the last chapter of this work, where some results are discussed that seemed to be important at the time of writing the first version. I would mention two of them which later turned out to be really important. The first one is the Nelson–Gross inequality which later played an important role in the theory of the so-called hypercontractive and logarithmic Soboliev inequalities. The second one is a method for construction of non-trivial self-similar fields worked out in a paper of Kesten and Spitzer. Several important limit theorems are based on the ideas of this paper. It is worth mentioning that it was Roland L’vovich Dobrushin who called my attention to these results, and he emphasized their importance. So I would like to finish this preface with a personal remark about him.

This work is the result of some joint research with Roland L’vovich Dobrushin. Although the book was written by me alone, Dobrushin’s influence is very strong in it. I have learned very much from him. It is rather difficult to explain what one could learn from him, because it was much more than just some results or mathematical arguments. There was something beyond it, some world view which is hard to explain. If I could give back something from what I had learned from him in this Lecture Note, then this would justify the work on it by itself.

Budapest, Hungary Péter Majer
15 August 2013

Preface to the First Edition

One of the most important problems in probability theory is the investigation of the limit distribution of partial sums of appropriately normalized random variables. The case where the random variables are independent is fairly well understood. Many results are known also in the case where independence is replaced by an appropriate mixing condition or some other "almost independence" property. Much less is known about the limit behaviour of partial sums of really dependent random variables. On the other hand, this case is becoming more and more important, not only in probability theory but also in some applications in statistical physics.

The problem about the asymptotic behaviour of partial sums of dependent random variables leads to the investigation of some very complicated transformations of probability measures. The classical methods of probability theory do not seem to work for this problem. On the other hand, although we are still very far from a satisfactory solution of this problem, we can already present some non-trivial results.

The so-called multiple Wiener–Itô integrals have proved to be a very useful tool in the investigation of this problem. The proofs of almost all rigorous results in this field are closely related to this technique. The notion of multiple Wiener–Itô integrals was worked out for the investigation of non-linear functionals over Gaussian fields. It is closely related to the so-called Wick polynomials which can be considered as the multi-dimensional generalization of Hermite polynomials. The notion of Wick polynomials and multiple Wiener–Itô integrals were worked out at the same time and independently of each other. Actually, we discuss a modified version of the multiple Wiener–Itô integrals in greatest detail. The technical changes needed in the definition of these modified integrals are not essential. On the other hand, these modified integrals are more appropriate for certain investigations, since they enable us to describe the action of shift transformations and to apply some sort of random Fourier analysis. There is also some connection between multiple Wiener–Itô integrals and the classical stochastic Itô integrals. The main difference between them is that in the first case deterministic functions are integrated, and in the second case so-called non-anticipating functionals. The consequence of this difference is that no technical difficulty arises when we want to define multiple

Wiener–Itô integrals in the multi-dimensional time case. On the other hand, a large class of non-linear functionals over Gaussian fields can be represented by means of multiple Wiener–Itô integrals.

In this work we are interested in limit problems for sums of dependent random variables. It is useful to consider this problem together with its continuous time version. The natural formulation of the continuous time version of this problem can be given by means of generalized random fields. Consequently we also have to discuss some questions about them.

I have not tried to formulate all the results in the most general form. My main goal was to work out the most important techniques needed in the investigation of such problems. This is the reason why the greatest part of this work deals with multiple Wiener–Itô integrals. I have tried to give a self-contained exposition of this subject and also to explain the motivation behind the results.

I had the opportunity to participate in the Dobrushin–Sinai seminar in Moscow. What I learned there was very useful also for the preparation of this Lecture Note. Therefore I would like to thank the members of this seminar for what I could learn from them, especially P.M. Bleher, R.L. Dobrushin and Ya.G. Sinai.

Contents

Acronyms

$\mathscr{D}$	The space of infinitely differentiable functions with compact support
$\mathscr{D}'$	The space of generalized functions on the space of test function $\mathscr{D}$
$d(A)$	The diameter of the set A
$\mathrm{Exp}\,\mathscr{H}_G$ and $\mathrm{Exp}\,\mathscr{K}_\mu$	The Fock space
$G(\cdot)$	The spectral measure of a stationary discrete or generalized random field
$G_n \stackrel{v}{\to} G_0$	The vague convergence of the locally finite measures G_n to the locally finite measure G_0
$H_n(x)$	The Hermite polynomial of order n with leading coefficient 1
$\mathscr{H}$	The (real) Hilbert space of square-integrable random variables measurable with respect to the σ-algebra generated by the random variables of a previously defined Gaussian random field
$\mathscr{H}_1$	The smallest subspace of the Hilbert space $\mathscr{H}$ containing the elements of the underlying Gaussian field
$\mathscr{H}_{\le n}$	The smallest subspace of the Hilbert space $\mathscr{H}$ containing the polynomials of order less than or equal to n of the random variables in the underlying Gaussian field
$\mathscr{H}_n$	The orthogonal completion of the subspace $\mathscr{H}_{\le n-1}$ in the Hilbert space $\mathscr{H}_{\le n}$
$h_\gamma(\cdot)$	The kernel function of the Wiener–Itô integral appearing in the diagram formula and depending on the diagram γ
$\bar{\mathscr{H}}_G^n$	The space of functions which can be the kernel function of an n-fold Wiener–Itô integral with respect to a random spectral measure Z_G with spectral measure G
$\mathscr{H}_G^n$	The subspace of $\bar{\mathscr{H}}_G^n$ consisting of symmetric functions
$\hat{\bar{\mathscr{H}}}{}_G^n$	The subspace of $\bar{\mathscr{H}}_G^n$ consisting of simple functions

$\hat{\mathscr{H}}_G^n$	The subspace of $\mathscr{H}_G^n$ consisting of simple (and symmetric) functions
$I_G(f_n)$	The normalized Wiener–Itô integral of the kernel function f_n of n variables with respect to the random spectral measure Z_G
$\bar{\mathscr{K}}_\mu^n$	The class of function which can be the kernel function of an n-fold Wiener–Itô integral with respect to a random orthogonal measure Z_μ
$\mathscr{K}_\mu^n$	The subspace of $\bar{\mathscr{K}}_\mu^n$ consisting of symmetric functions
$\hat{\bar{\mathscr{K}}}_\mu^n$	The set of simple functions appearing in the definition of n-fold Wiener–Itô integrals with respect to a random orthogonal measure Z_μ
$\mathscr{K}$	The Hilbert space of square integrable random variables measurable with respect to the σ-algebra generated by the random variables $Z_\mu(A)$ of a random orthogonal measure Z_μ
$\mathscr{K}_{\le n}$	The subspace of $\mathscr{K}$ generated by the polynomials of the random variables $Z_\mu(A)$ of the orthogonal random field Z_μ which have order less than or equal to n
$\mathscr{K}_n$	The orthogonal completion of the subspace $\mathscr{K}_{\le n-1}$ in the Hilbert space $\mathscr{K}_{\le n}$
$:P(\xi_1,\dots,\xi_n):$	The Wick polynomial corresponding to the polynomial $P(x_1,\dots,x_n)$ and Gaussian random vector $(\xi_1,\dots,\xi_n)$
$\mathscr{S}$	The class of test functions in the Schwartz space
$\mathscr{S}^c$	The class of complex number valued test functions in the Schwartz space
$\mathscr{S}'$	The Schwartz space of generalized functions
$S_{\nu-1}$	The ν-dimensional unit sphere
Sym f	The symmetrization of the function f
T_m and T_t	The shift operator with parameter $m \in \mathbb{Z}_\nu$ and $t \in R^\nu$
$X(\varphi)$	The value of the generalized field $X(\cdot)$ at the test function φ
$Z_G(\cdot)$	The (Gaussian) random spectral measure corresponding to the spectral measure G
$Z(\,dx)$	The (Gaussian) random spectral measure whose spectral measure is $\frac{1}{2\pi}$ times the Lebesgue measure on $[-\pi,\pi)$
$Z_\mu(\cdot)$	The random orthogonal measure corresponding to the measure μ
$\mathbb{Z}_\nu$	The set of lattice points in the ν-dimensional space with integer coordinates
$\Gamma(n_1,\dots,n_m)$	The space of diagrams in the diagram formula
$\bar{\Gamma}$	The space of closed diagrams
$\lvert\gamma\rvert$	The number of edges in a diagram γ
$\mu_n \xrightarrow{w} \mu$	The weak convergence of the probability measures μ_n to the probability measure μ

$\xi_N \xrightarrow{\mathscr{D}} \xi_0$	The convergence of the random variables ξ_N to the random variable ξ_0 in distribution, i.e. the weak convergence of the distributions of ξ_N to the distribution of ξ_0
Π_n	The group of permutations of the set $\{1, \dots, n\}$
$\rho_p(\mu, \nu)$	The Prokhorov metric of the probability measures μ and ν
$\chi_A(\cdot)$	The indicator function of the set A.
$\tilde{\chi}_n(x)$	The Fourier transform of the indicator function of the unit cube $\prod_{p=1}^{\nu} [n^{(p)}, n^{(p)} + 1)$, where $n = (n^{(1)}, \dots, n^{(p)})$
$\ominus$	The orthogonal completion of a subspace of a Hilbert space
$\tilde{}$	Fourier transform
$*$	Convolutions
$\stackrel{\Delta}{=}$	Identity in distribution
$\Rightarrow$	Stochastic convergence
$\int'$	Wiener–Itô integral with respect to a random orthogonal measure
$[x]$	Integer part of a real number x