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Modal Interval Analysis

New Tools for Numerical Information

Springer

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Preface

The basic idea of Interval Mathematics is that ordinary set-theoretical intervals $I(\mathbb{R})$ provide a consistent support for numerical computing. The set-theoretical form of Interval Analysis has produced a large amount of work since its initiation in the late 1950s, [1, 60, 61]. There are some later papers devoted to the structural analysis of the method or its completion [13, 14, 49, 66, 72, 87, 96, 97], but with no fundamental departure from its initial set-theoretical foundations.

This book presents a new interval theory, the Modal Interval Analysis (MIA), as a structural, algebraic, and logical completion of the classical intervals. The starting point of MIA is quite simple: to define a modal interval attaching a quantifier to a classical interval, and to introduce the basic relation of inclusion between modal intervals by means of the inclusion between the sets of predicates they accept. So a modal interval consists in a classical interval, which defines its domain, and a quantifier, which defines its modality. This modal approach introduces interval extensions of the real continuous functions, gives equivalences between logical formulas and interval inclusions, and provides the semantic theorems that justify these equivalences and guidelines to get these inclusions.

The significant change of perspective in the treatment of information, coming from this new approach, makes Modal Interval Analysis more a new tool for the general practice of Numerical Applied Mathematics than a contribution to the previous Interval Theory. It supposes a complete philosophy of numerical information which is, or can be, its best virtue and produces at each stage of its development not only one body of solutions, but also questions leading to the construction of the next stage of the theory. Modal intervals system is not a breaking-off with the classic intervals, but a algebraic, structural, and logic completion of them that opens a way to new forms of numerical information treatment.

This book summarizes the most relevant results and features of MIA and also provides several application examples that illustrate the use of them in different problems and domains. The book contains the detailed development of the theory, the main concepts and results, together with several examples to clarify their meaning and to balance the mathematical items and proofs. Definitions contain the concepts, theorems provide the main results, and corollaries and lemmas embrace detailed logical developments.

Implementation of arithmetics, computation rules, and some algorithms is in the software MISO developed by the research group MiceLab of the University of Girona (Spain) and available at http://www.cs.utep.edu/interval-comp/intsoft.html

After the introductory Chap. 1, about real, digital numbers and intervals, and limitations of classical interval theory, Chap. 2 gives an account of the fundamental definitions and structures which support the semantically oriented system of modal intervals $I^*(\mathbb{R})$. Basic concepts such as predicates, canonical coordinates, modal inclusion, and equality, duality, interval predicates and co-predicates, rounding, and lattice operators are presented in detail. The set $I^*(\mathbb{R})$ of modal intervals turns out to be a completion of $I(\mathbb{R})$, in a similar way to that in which the complex numbers are a completion of the real numbers. So, a subset of $I^*(\mathbb{R})$, the "proper" modal interval ($[a, b]$ with $a \leq b$) is identifiable with a classical interval $[a, b]$ and all the results of Classical Interval Analysis are also results of Modal Interval Analysis.

In accordance with the sense of the term *analysis* in Mathematics, as a discipline in which the objects of study are, first and foremost, functions, Modal Interval theory can be considered, indeed, as an analysis because it studies a numerical field, the modal intervals, and the functions defined on it. So, Chap. 3 deals with the interval extension of the real continuous functions. The historic reason for the theory of these extensions of continuous real functions is to overcome the limited character of the classical set-theoretical approach. The geometrical semantics of $(n+1)$-dimensional real space, $\mathbb{R}^{n+1}$, is basically defined by the continuous functions f from $\mathbb{R}^n$ to $\mathbb{R}$. The semantic interval functions f^* and f^{**} from $I^*(\mathbb{R}^n)$ to $I^*(\mathbb{R})$, consistently referring to the continuous functions f from $\mathbb{R}^n$ to $\mathbb{R}$, are obtained by translating to modal terms the set-theoretical definition of a simple interval extension of a real continuous function. When the continuous real function is considered as a syntactic tree, it can also be extended to a rational interval function fR from $\mathbb{R}^n$ to $\mathbb{R}$, by using the computing program implicitly defined by the syntax of the expression defining the function. The idea of *interpretability* is given as a definite formulation via the cornerstones which are the *Semantic Theorems.*

Chapter 4 is devoted to characterizing the existence of optimal computations for a semantic function. Both modal extensions f^* and f^{**} are semantically interpretable, but not computable in general. When f^* and f^{**} are computed through the modal rational extension fR, a null, partial, or complete loss of information is generated. The point is to find functions for which the program fR is optimal, that is, for which $fR(X)$ equals $f^*(X)$ and $f^{**}(X)$.

Modal interval arithmetic operators and metric functions are considered in Chap. 5. The modal arithmetic coincides, certainly, with the arithmetic of the Kaucher's Extended Interval Space $\mathbb{IR}$ [49] with an important difference: in $I^*(\mathbb{R})$ interval results provided by the arithmetic have a logical meaning related to the points of the operand intervals domains, thanks to the semantic and interpretability theorems. Thus, unlike Extended Interval Space, which is a formal and algebraic completion of the Classic Intervals Space $I(\mathbb{R})$, Gardenyes' Modal Intervals are also a semantic completion of $I(\mathbb{R})$.

Chapter 6 contains procedures for solving interval linear equations and systems. The Jacobi method is adapted to interval systems together with convergence and non-convergence conditions. An important point is to provide a logical meaning to the solution using the semantic theorems.

The definition of the semantic extension of a real continuous function does not provide any indication about how to compute it. Some conditions under which f^* can be computed through the syntactic extension are given in Chap. 3, but in the most general case it is obtained by means of an algorithm developed in Chap. 7, referred as *f^*-algorithm*. First, some considerations about twins (intervals of intervals) are given to provide a background for this f^*-algorithm.

The matter of the necessary rounding is introduced in Chap. 1 and dealt with in the following chapters. Nevertheless, a shortcoming of the modal theory is managing the rounding of an interval when it appears both as it is and dualized in the same computation, for example in the solution of a linear system. To overcome this difficulty, in Chap. 8 a new object based on modal intervals is introduced: marks. Definitions, relations, the extension of a continuous function to a function of marks, operators of marks, and the corresponding semantic results are given in detail together with examples, not only to illustrate the different concepts and results, but also to show that marks can be used in a very practical way to aware about ill computations which can appear in the use of algorithms with real numbers.

Chapter 9 closes the loop opened in Chap. 2 dealing with intervals and modal intervals of marks, following a parallel development to the one started in Chap. 2 for modal intervals of real numbers $I^*(\mathbb{R})$. Predicates, relations, lattice operators, semantic and syntactic functional extensions to intervals of marks, and the semantic theorem, together with the arithmetic operators, are outlined throughout the chapter.

Finally, Chap. 10 is devoted to showing some applications of modal intervals. Specifically, they are used to deal with three problems: minimax, characterization of solution sets of quantified constraint satisfaction problems, and statement of problems in control engineering or process control from a semantic point of view. Algorithms and procedures about these topics are presented, together with examples to illustrate the procedures.

The beginnings of MIA can be situated in the SIGLA project, developed at the University of Barcelona in the late 1970s. In the 1980s and 1990s it was further continued by the SIGLA/X group (University of Barcelona and Polytechnical University of Catalonia in Spain), some of whose results can be found in [20–27, 86, 90]. The kernel of its main applications has been developed inside the MiceLab of the University of Girona (Spain) from the 1990s to the present.

We, the authors, are indebted to many people who have played significant roles in the development of Modal Interval Theory. First of all, to Dr. E. Gardenyes, founder of the Modal Interval Analysis since his first works in the 1980s until early 2000s together with a set of coworkers, Dr. H. Mielgo, Dr. A. Trepat, Dr. J.M. Janer, Dra. R. Estela, and some of us. With this book we want to render him tribute. Along the hardcore of the theory, we have wanted to preserve, in some way, his conceptualist style and notation, except for some adaptations to the standards. Also, we wish to thank several colleagues for their valuable comments and criticisms, Dr. V. Kreinovich, Dr. A. Neumaier, Dr. L. Jaulin, Dr. A. Goldsztein, Dr. S. Ratschan and, in a very special way, we thank Dr. S.P. Shary and Dr. E. Walter, for the patient reading of the manuscript. Nevertheless, any error, omission, or obscurity are entirely our responsibility.

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Notations

In order to make clear enough the main mathematical subjects put to work along the text, we have used the following typefaces and notations:

- Lowercase and italic x for a *real number*.
- Lowercase and italic f for a *real function of one variable*.
- Lowercase, bold and italic $\boldsymbol{x}$ for a *vector* with real components

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_m).$$

- Lowercase, bold and italic $\boldsymbol{f}$ for a *vectorial function* with functional components

$$\boldsymbol{f} = (f_1, f_2, \ldots, f_m).$$

- Uppercase, italic A and apostrophe for a *classical interval* with real bounds

$$A' = [a_1, a_2]' \text{ or } A' = [\underline{a}, \overline{a}]'.$$

- Uppercase and italic A for a *modal interval* with real bounds

$$A = [a_1, a_2] \text{ or } A = [\underline{a}, \overline{a}].$$

- Uppercase, bold and italic for an *modal interval vector* with modal interval components

$$\boldsymbol{A} = (A_1, A_2, \ldots, A_m).$$

- Uppercase and bold for a *real matrix* with real elements

$$\mathbf{A} = (a_{ij}).$$

- Also uppercase, bold and italic for an *interval matrix* with interval elements

$$\boldsymbol{A} = (A_{ij}).$$

The context prevents any lack of distinction between interval vector and interval matrix, which is often irrelevant because a vector in a finite-dimension vectorial space can be identified with a row or column matrix.

- $\mathrm{wid}(X)$ for the *width*, $\mathrm{mid}(X)$ for the *midpoint*, $\mathrm{mig}(X)$ for the *mignitude* and $\mathrm{abs}(X)$ for the *absolute value* of an interval X
- $|X|$ for the interval *absolute value function*.
- $\mathrm{dist}(X, Y)$ for the *Hausdorff distance* between two intervals X and Y.
- $\mathrm{Q}(x, X)$ for the *modal quantifier*.
- Lowercase mathfrak $\mathfrak{m}$ for a *mark* with real attributes

$$\mathfrak{m} = \langle c, t, g, n, b \rangle.$$

- Uppercase mathfrak and apostrophe $\mathfrak{A}'$ for a *set-theoretical interval of marks* with marks bounds

$$\mathfrak{A}' = [\underline{\mathfrak{a}}, \overline{\mathfrak{a}}]'.$$

- Uppercase mathfrak $\mathfrak{A}$ for a *modal interval of marks* with marks bounds

$$\mathfrak{A} = [\underline{\mathfrak{a}}, \overline{\mathfrak{a}}].$$

- $\mathbb{A}$ for a *twin* with interval bounds

$$\mathbb{A} = |[\underline{A}, \overline{A}]|.$$

- $\mathbb{R}$ for the *set of real numbers*.
- $I(\mathbb{R})$ for the *set of classical intervals*, as subsets of $\mathbb{R}$.
- $I^*(\mathbb{R})$ for the *set of modal intervals*.
- $\mathbb{M}(t, n, b)$ for the *set of marks* with tolerance t, number of digits n and scale basis b.
- $I(\mathbb{M}(t, n))$ for the *set of proper intervals of marks*, abridged to $I(\mathbb{M})$ when the type of the marks is arranged in advance.
- $I^*(\mathbb{M}(t, n))$ for the *set of modal intervals of marks*, abridged to $I^*(\mathbb{M})$ when the type of the marks is arranged in advance.
- $I(I^*(\mathbb{R}))$ for the *set of proper twins*.
- $I^*(I^*(\mathbb{R}))$ for the *set of twins*.

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