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Hypoelliptic Laplacian and Bott–Chern Cohomology

A Theorem of Riemann–Roch–Grothendieck in Complex Geometry

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Le poumon, le poumon, vous dis-je.

MOLIÈRE
Le malade imaginaire

Preface

The purpose of this preliminary chapter is to give a general and simple introduction to some of the ideas and techniques that are used in this book.

We prove a version of a Riemann-Roch-Grothendieck theorem, in the context of complex Hermitian geometry. Our result sits at the crossroads of algebraic geometry, of which the theorem of Riemann-Roch-Grothendieck is a crucial element, and of analysis, in which the Atiyah-Singer index theorem for elliptic operators is an analytic and topological counterpart to the theorem of Riemann-Roch-Grothendieck.

Our approach to the problem at hand is mostly analytical. We will try to explain in elementary terms what methods have been traditionally relevant in the index theory of elliptic operators, and also to introduce the ideas that are used in the solution of the problem, in particular by the introduction of hypoelliptic operators.

The index theorem of Atiyah-Singer [AS68a, AS68b] gives a formula for the index of an elliptic pseudodifferential operator on a smooth compact manifold. It expresses the index as the integral of characteristic classes associated with the principal symbol of the elliptic operator. Atiyah and Singer have shown that it is enough to prove the index theorem for a specific class of elliptic operators, the Dirac operators on Riemannian manifolds, which are first-order elliptic differential operators. The principal symbol of the square of such operators is scalar, and coincides with the principal symbol of minus the Laplacian. For Dirac operators, the index theorem can be proved by the so-called heat equation method. This method works only for a specific class of Dirac operators that are associated with the Levi-Civita connection on the considered Riemannian manifold. We will call such operators *classical Dirac operators*. Since the index only depends on the principal symbol, it is enough to prove the index theorem for the classical Dirac operators.

For classical Dirac operators, the heat equation method is based on the mechanism known as the 'fantastic cancellations' that were conjectured by McKean and Singer [MS67], and established by Gilkey [Gi73] and Atiyah-Bott-Patodi [ABP73] by algebraic arguments. The *local index theorem* establishes the fact that the difference of traces of heat kernels associated with the square of the classical Dirac operator on the diagonal, instead of being singular as time tends to zero, is in fact non-singular, and the constant term can be explicitly computed in terms of characteristic forms in Chern-Weil theory. The index theorem for Dirac operators follows from the local index theorem.

Alvarez-Gaumé [AG83] and other physicists have provided a crucial input in viewing the 'fantastic cancellations' not as a wonderful accident, but as a consequence of hidden properties of the Dirac operator, known in physics as supersymmetry. A proof of the local index theorem based on supersymmetry was provided by Getzler [Ge86, BeGeV92], as well as other proofs [B84a, B84b] based on probabilistic arguments.

In previous work [B89], we characterized the Dirac operators associated with connections that differ from the Levi-Civita connection, for which 'fantastic cancellations' still occur, in terms of the torsion of these connections. Let us here briefly mention that the square of a Dirac operator is the sum of an operator closely related to minus the Laplacian and of an operator of order 0 that is a matrix term. This matrix term controls the 'fantastic cancellations' in local index theory. Our results just mean that this 0-order term is suitably controlled.

The above considerations are especially relevant in the case of complex Hermitian manifolds. If the metric on the manifold is Kähler, the complex Dirac operator obtained by taking the sum of the Dolbeault operator and its adjoint is a classical Dirac operator. In this case, the local index theorem gives the formula of Riemann-Roch Hirzebruch for the Euler characteristic of a holomorphic vector bundle. Even if the manifold cannot be equipped with a Kähler metric, deforming the complex Dirac operator to a classical Dirac operator still gives a Riemann-Roch Hirzebruch formula. In [B89], we showed that if the Kähler form is $\overline{\partial}\partial$-closed, there is a local index theorem for the associated complex Dirac operator.

Let us now consider a proper fibration, where the fibres are compact manifolds carrying a family of elliptic pseudodifferential operators. To this family, one can associate a topological invariant on the base, the index bundle of the family. The families index theorem of Atiyah-Singer [AS71] gives a formula for the index bundle. The Chern character of the index bundle is an even cohomology class on the base, the ordinary index of the Dirac operator along the fibre being the term of degree 0.

In [B86a], we gave a local index theoretic proof of the Chern character version of the families index theorem of Atiyah-Singer for Dirac operators. The local index theorem for families is a refined version of the families index theorem, in the same way as the local index theorem refines on the index theorem. It consists in replacing cohomology classes by closed differential forms on the base. Quillen's superconnections [Q85] allowed us to properly formulate this refined version of the families index theorem, and to produce an object canonically associated with the geometry of the problem, the Levi-Civita superconnection of the fibration that also exhibits 'fantastic cancellations'.

Let us explain in elementary terms how one can think of the local families index theorem from the point of view of *adiabatic limits.* Consider a Riemannian metric on the source of the submersion. Add to this metric the pull back of a metric of the base that is scaled by a factor $1/\epsilon$. Making ϵ tend to 0 is also called passing to the adiabatic limit. As $\epsilon \to 0$, the fibres get further and further apart. The corresponding Laplacian of the total space converges to the Laplacian of the fibre, and the classical Dirac operator of the total space converges to the Dirac operator of the fibre. For each $\epsilon > 0$, there is a corresponding local index theorem. As explained in [B98, section 2], the local families index theorem is just the limit as $\epsilon \to 0$ of the local index theorem for a given ϵ. The fact that the limit exists is not obvious. For instance, one can show that as $\epsilon \to 0$, the Levi-Civita connection on the total tangent bundle converges. More generally, the Levi-Civita superconnection is the *proper limit* as $\epsilon \to 0$ of the corresponding classical Dirac operators on the total space. As in the case of the local index theorem for classical Dirac operators, the proof of the local families index theorem is possible because we have a suitable control on the degree 0 term in a family of fibrewise elliptic differential operators.

From a cohomological point of view, nothing has been gained from the local families index theorem. However, if we want to obtain explicit differential forms representing the Chern character, or to construct connections and metrics, this result is the starting point of the theory of determinant bundles [BF86] and Quillen metrics [Q85a, BGS88c].

Let us now consider the case where the submersion is a holomorphic map of complex manifolds. Assume that the total space is equipped with a Hermitian metric, so that the fibres carry a corresponding family of complex Dirac operators. From the point of view of the families index theorem of Atiyah-Singer, since this family can be deformed in the smooth category to the corresponding family of classical Dirac operators, the families index theorem for this family is already known. If the metric on the total space is Kähler, the family of complex Dirac operators is just a family of classical Dirac operators, to which the local families index theorem still applies. In this book, we show that if the Kähler form of the Hermitian metric on the total space is $\overline{\partial}\partial$-closed, there is still a local form of the families index theorem, this last result being just an adiabatic limit of the results described above in the case of one single manifold.

The purpose of the book is to prove a Riemann-Roch-Grothendieck theorem taking values in a refinement of the cohomology of the base, its Bott-Chern cohomology. This cohomology takes into account the (p, q)-grading on smooth differential forms, and refines on de Rham cohomology. If the Kähler form on the total space is closed, from results of [BGS88b, BK92], the local families index theorem of [B86a] is enough to prove the result we seek. In the book, we show by an adiabatic limit argument that this is still the case if the Kähler form on the total space is $\overline{\partial}\partial$-closed. If no such Kähler form exists, whatever proof there is breaks down. In particular, the deformation argument of the families of complex Dirac operators to the family of classical Dirac operators becomes irrelevant, because while this deformation preserves the information contained in de Rham cohomology of the base, it destroys the information contained in its Bott-Chern cohomology.

The reason why a local index theoretic proof of the families index theorem for the family of complex Dirac operators also breaks down is that if ω is the Kähler form on the total space, $\overline{\partial}\partial\omega$ appears explicitly in the zero-order term we mentioned before, and that such a term is, for reasons it would take too long to explain, a *bad term*. It is a similar term which explains why there is no standard local index theorem for a given complex manifold when $\overline{\partial}\partial\omega$ does not vanish. In this last situation, the Kähler form ω is responsible for the appearance in the formula for the square of the complex Dirac operator of a Laplacian-like operator, which is 'good', and of the *bad term* $\overline{\partial}\partial\omega$. One can then say that like everything else, ellipticity, while being good from a certain point of view, is bad from another point of view.

The present book is an attempt to get out of this quagmire. If ellipticity is responsible for the failure of the proof of the local index theorem and of the local families index theorem, it should be abandoned. More precisely, we have at the same time to manage to produce a family of operators along the fibre that contains operators 'almost as good' as the family of fibrewise-like elliptic Laplacians–such operators will be said to be *hypoelliptic*– while hoping that whatever replaces the 'bad term' $\overline{\partial}\partial\omega$ will ultimately behave itself, that is become 'good'. Needless to say, contrary to the smooth deformation from complex Dirac operators to classical Dirac operators, the deformation of our initial family of operators to the new family has to be done in such a way as to preserve what we want to compute in the first place, a cohomology class in Bott-Chern cohomology.

To explain in simple terms what the solution consists of, let us go back to the idea of adiabatic limit. We already explained that the local families index theorem is an adiabatic limit of the local index theorem. Along the same line, one can expect that if one can indeed obtain a 'good' version of the local index theorem for complex non-Kähler manifolds, it will be possible to ultimately produce the right family of operators in the context of families. However, we already explained that for one single manifold, the deformation of the complex Dirac operator to the classical Dirac operator produced a smooth version of the local index theorem. Why is this deformation not acceptable in our context? It is because it deforms the $\overline{\partial}$ operator itself, while such a deformation is ultimately forbidden by what we want to prove.

The solution consists of several steps. Again, we consider the case of a single compact Hermitian manifold X. First we replace X by $\mathcal{X}$, the total space of its tangent bundle. By viewing X as the zero section of $\mathcal{X}$, we construct on $\mathcal{X}$ a Koszul complex, which is quasi-isomorphic to the original Dolbeault complex on X. On $\mathcal{X}$, we will then produce an *exotic* version of Hodge theory that is associated not with a Hermitian product, like classical Hodge theory, but with a Hermitian form of signature (∞, ∞). This Hermitian form is essentially like Serre duality on the base X, and an Hermitian duality along the fibre $\widehat{TX}$, where the obvious canonical antipodal involution of $\mathcal{X}$ is taken into account. Two distinct metrics appear in the construction, the metric on the base X and its Kähler form ω, and also the metric along the fibre $\widehat{TX}$. The resulting Hodge-like Laplacian will contain minus the fibrewise Laplacian, and a vector field on $\mathcal{X}$ which is essentially the generator of the geodesic flow on $\mathcal{X}$ for the metric along the fibre. By a result of Hörmander [H67], the sum of these two terms is hypoelliptic. The Hodge-like Laplacian, also contains a nonnegative potential of the form $|Y|^4$, and also $|Y|^2 \overline{\partial}\partial\omega$, which will now turn out to be 'good'. This hypoelliptic Laplacian is the main tool that is used in the proof of our main result. For other aspects of the theory of the hypoelliptic Laplacian, we refer to [B12].

Of course, the above short description of the exotic Hodge-like Laplacian leaves several questions out:

1. Why do such operators deform the ordinary Hodge Laplacian on X?
2. Why does this construction give an acceptable deformation of the original Hodge Laplacian?
3. Why has the 'bad' term $\overline{\partial}\partial\omega$ become 'good', when acquiring the extra weight $|Y|^2$?

If you want to know the answer to these questions, read the book!

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Contents