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Sub-Riemannian Geometry

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Preface

Following a suggestion by Héctor J. Sussmann we organized, in the summer of 1992 in Paris, a satellite meeting of the first European Congress of Mathematics. The topic of the meeting was "Nonholonomy", and officially titled:

JOURNÉES NONHOLONOMES
Géométrie sous-riemannienne, théorie du contrôle, robotique

It was held at Université Paris VI–Pierre et Marie Curie (Jussieu), on June 30th and July 1st, 1992.

Sub-Riemannian Geometry (also known as Carnot Geometry in France, and Nonholonomic Riemannian Geometry in Russia) has been a fully-fledged research domain for fifteen years, with motivations and ramifications in several parts of pure and applied mathematics, namely:

- Control Theory;
- Classical Mechanics;
- Riemannian Geometry (of which Sub-Riemannian Geometry constitutes a natural generalization, and where sub-Riemannian metrics may appear as limit cases);
- Gauge theories;
- Diffusions on manifolds;
- Analysis of hypoelliptic operators; and
- Cauchy-Riemann (or CR) Geometry.

Although links between these domains had been foreseen by many authors in the past, it is only in recent years that Sub-Riemannian Geometry has been recognized as a possible common framework for all these topics (e.g., the conference paper by Agrachev at the 1994 International Mathematical Congress in Zurich).

To illustrate this fact, it should be noted that the first editor of this volume was interested in nonholonomy, following encouragement by Robert Azencott to provide a geometric frame for the study of non-elliptic diffusions. The second editor, a specialist in real algebraic geometry, came to the same subject after a collaboration with roboticians, when it became clear that nonholonomy was one of the main problems in robotics.

The first article, by André Bellaïche, is a local study of sub-Riemannian structures. After some general definitions, he examines the notion of tangent space at a point of a sub-Riemannian manifold: this space has a natural structure of nilpotent Lie group with dilations at regular points, and of a quotient of such a group otherwise, showing some similarity with the Riemannian case where the tangent space is a linear space, i.e., a commutative Lie group with dilations.

The next and rather extensive article by M. Gromov is impossible to summarize in few words. It builds on basic facts, given in the preceding paper, and then adresses an impressive number of questions and conjectures, in general inspired by Riemannian geometry. The point of view of Gromov is roughly the following: if one "lives" inside a metric space, whose metric comes from a sub-Riemannian structure on a manifold, what are the properties of the distribution (differentiable structure, dimension of the distribution, dimension of its derived distributions, etc.) that one can recover?

The paper by R. Montgomery describes the phenomenon of abnormal extremals (or abnormal geodesics) by surveying their properties, and by describing the first example (due to the author) of such a (minimizing) geodesic.

This subject is also taken up by H. Sussmann, who here gives yet further exhaustive examples in dimension 4, showing that the phenomenon of abnormal extremals is "generic" in sub-Riemannian geometry.

Finally, J.-M. Coron's paper takes a different approach, since he deals with stabilization and feedback laws, closer to control theory than sub-Riemannian geometry. Nevertheless, the reader will find this paper to be remarkably consistent with the previous ones.

The interest and coherence of the conference papers induced us to bring these texts together in the present volume. Publication comes late—for which we apologize—but we hope the reader will find the waiting worthwhile.

We thank the five authors for the confidence they have shown in this project.

We thank Héctor Sussmann for initiating the "Journées" and Jean-Paul Laumond, from the LAAS in Toulouse, who awakened or, better, renewed our interest in these questions, and who also was one of the initiators of this meeting. We thank also Birkhäuser for publishing these texts.

André Bellaïche, Jean-Jacques Risler

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