Part III
Extension, Stability, and Stabilization of Weak Solutions

Introduction to Part III

Fix an arbitrary $0 < T < +\infty$.

In Section 7 of Chapter 6 we studied the following question:

1. What must be c.n.o. A and B in H for every bounded weak solution of equation (1) on $(0, T]$ to have a limit in H as $t \to 0$?

We have answered this question in the special case that $P = \sigma(A, B)$ satisfies (2).

This additional assumption is not essential for the resolution of the problem. In Chapter 8, we waive this assumption and answer this question in the general case where A and B are arbitrary c.n.o. in H.

To this end, we consider c.n.o. A and B in H in a more general case than in Section 7 of Chapter 6. Namely, we consider the case where (3.3) (or, what is the same, (2.2)) holds but (3.4) may not be valid. In this more general case, we obtain the general form of weak solutions of equation (1) on $(0, T]$ which have a limit in H as $t \to 0$, as well as the general form of bounded weak solutions of (1) on $(0, T]$. As well, we obtain the general form and the set of initial data of weak solutions of equation (1) on $[0, T]$.

Chapter 7 is devoted entirely to solution of these problems.

Further treatments are based on these results.

First, this enables us to answer the question 1. As well, we answer the following question:

What must be c.n.o. A and B in H for every bounded weak solution of equation (1) on $[0, T)$ to have a limit in H as $t \to T$?

These questions are answered in Chapter 8.

The last question may be considered as one of a series of questions relating to behaviour as $t \to T$ and extension of weak solutions of equation (1) on a finite interval $[0, T)$. Let us give some of these questions.

What must be c.n.o. A and B in H for every weak solution of (1) on $[0, T)$ to be bounded?

What must be c.n.o. A and B in H for every weak solution of (1) on $[0, T)$ to have a limit in H as $t \to T$?

What must be c.n.o. A and B in H for every weak solution of (1) on $[0, T)$ to be extendable up to a weak solution of (1) on the whole R_+?

These and similar questions are answered in Chapter 9.

In Chapter 10, we pose similar questions where $T = +\infty$. Let's give some of these questions.

What must be c.n.o. A and B in H for every weak solution of (1) on $R_+ = [0, +\infty)$ to be bounded?

What must be c.n.o. A and B in H for every weak solution of (1) on R_+ to have a limit in H as $t \to +\infty$?

These and other similar questions are answered in Chapter 10. In fact, of concern are problems of stability, stabilization of weak solutions, asymptotic stability, exponential stability of equation (1).

For ordinary linear differential equations in a finite-dimensional Hilbert space, these problems were studied beginning, apparently, from A.M. Lyapunov. For first order linear differential equations in Banach spaces

$$y'(t) + Ay(t) = 0,$$

these problems have been studied extensively during the last decades (see, for instance, [57, 138]).

In Chapter 10, we obtain criteria for stability, stabilization of weak solutions, asymptotic stability, exponential stability on $R_+ = [0, +\infty)$ for the equation

$$y''(t) + Ay'(t) + By(t) = 0$$

where A and B are arbitrary commuting normal operators in a Hilbert space H.

We consider all these questions in the «weak» setting, i.e., for weak solutions. Similarly, one can do this for usual solutions, with $\| y(t) \|$ replaced by $\max_{i=0,1,2} \| y^{(i)}(t) \|$. We omit the corresponding statements.