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Floquet Theory for Partial Differential Equations

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To my Teacher, Professor Selim Krein

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Introduction

Linear differential equations with periodic coefficients constitute a well developed part of the theory of ordinary differential equations [17, 94, 156, 177, 178, 272, 389]. They arise in many physical and technical applications [177, 178, 272]. A new wave of interest in this subject has been stimulated during the last two decades by the development of the inverse scattering method for integration of nonlinear differential equations. This has led to significant progress in this traditional area [27, 71, 72, 111-119, 250, 276, 277, 284, 286, 287, 312, 313, 337, 349, 354, 392, 393, 403, 404]. At the same time, many theoretical and applied problems lead to periodic partial differential equations. We can mention, for instance, quantum mechanics [14, 18, 40, 54, 60, 91, 92, 107, 123, 157-160, 192, 193, 204, 315, 367, 412, 414, 415, 417], hydrodynamics [179, 180], elasticity theory [395], the theory of guided waves [87-89, 208, 300], homogenization theory [29, 41, 348], direct and inverse scattering [175, 206, 216, 314, 388, 406-408], parametric resonance theory [122, 178], and spectral theory and spectral geometry [103-105, 381, 382, 389].

There is a significant distinction between the cases of ordinary and partial differential periodic equations. The main tool of the theory of periodic ordinary differential equations is the so-called Floquet theory [17, 94, 120, 156, 177, 267, 272, 389]. Its central result is the following theorem (sometimes called Floquet-Lyapunov theorem) [120, 267]. Consider a system

$$\frac{du}{dt} = A(t)u \tag{1}$$

with a periodic matrix coefficient $A(t)$. (We shall assume that, modulo a change of the variable, the period is 1.) Then there is a substitution $v = B(t)u$ with an invertible 1-periodic matrix $B(t)$ that converts (1) into a system with a constant matrix coefficient A_0. By the famous L. Euler's theorem [106], any solution of such a system is a linear combination of the exponential-polynomial solutions:

$$v(t) = e^{i\lambda t} \left(\sum_{k=0}^{n} a_k t^k \right).$$

Returning to the original unknown function, we obtain an expansion of all solutions of (1) into the solutions of the form

$$u(t) = e^{i\lambda t} \left(\sum_{k=0}^{n} a_k(t)\, t^k \right). \tag{2}$$

Here the vector valued functions $a_k(t)$ are 1-periodic. Solutions (2) are usually called Floquet solutions. The non-zero complex number $z = exp\ i\lambda$ is the Floquet exponent (or Floquet multiplier) of the solution. The complex number λ is said to be the quasi-momentum of the solution. (This name has come from quantum theory of solids [18, 414].) Classical results [17, 94, 177, 178, 272, 389] show that various properties of equation (1) can be deduced from the distribution of Floquet exponents: stability, solvability of non-homogeneous equations, exponential dichotomy, spectral theory, etc.

There are two standard ways of proving the Floquet theorem. One of them is to find the aforementioned periodic substitution $B(t)$, and to apply Euler's theorem to the resulting constant coefficient equation. Another approach is to consider the monodromy operator, i.e. the shift along the trajectories of the system onto the period. The Floquet expansion of solutions corresponds to the Jordan representation of this operator. If one tries to carry over these two methods to periodic partial differential equations, one will face major difficulties. First of all, it is impossible in general to convert a partial differential equation that is periodic with respect to several variables into an equation with constant coefficients. Secondly, the monodromy operator can be correctly defined only for time-periodic evolution equations under the assumption that the corresponding Cauchy problem is correctly posed. This approach cannot be used, for instance, in the case of a stationary Schrödinger operator with a potential that is periodic in several variables. Even in the case of time-periodic evolution equations the approaches analogous to those applied to ordinary differential equations do not always work. In order to develop the Floquet theory for periodic partial differential equations, we need a new point of view; some significant progress has been made during the last decade. My main goal is to give an up to date account of this theory.

My approach to the Floquet theory is analogous to the one adopted in [99, 273, 274, 324] for the case of constant coefficient partial differential equations. The results and technique of V. Palamodov [321-326, 421] play an especially important role.

Let me describe briefly the contents of the book.

Chapter 1 contains some preliminary information on functional analysis, operator theory and complex analysis. The main results that will play the major role in the following chapters are concentrated in sections 1.7 and 1.8. They are rather technical, and I recommend to the reader who is interested mainly in differential equations to skip chapter 1 on the first reading. The statements of the main results of chapters 2-6 can be understood without referring to sections 1.7 and 1.8. The proofs, however, depend strongly on chapter 1.

Chapter 2 provides the definitions and properties of transforms that play the role of the Fourier transform.

Chapter 3 is devoted to the principal results of the Floquet theory for hypoelliptic equations.

In chapter 4 it is shown how the Floquet theory is related to other properties of periodic equations (for instance, to spectral theory).

Chapter 5 treats in more detail the case of evolution equations of hypoelliptic (in particular, parabolic) type. Writing this chapter, I found out that it was impossible to cover in one chapter even the case of time-periodic parabolic equations. I plan to give a more complete account of this topic in another publication.

Chapter 6 contains a brief consideration of related problems: equations with deviating arguments, more general groups of periods, etc.

Every chapter has a final section devoted to additional comments and references.

I would like to mention that the main emphasis of this book is on the Floquet theory, so many related topics (spectral problems, positive solutions, non-commutative groups of periods, etc.) are just touched on. Some of them (like non-commutative Floquet theory) are still in the beginning stages of development. My presentation of these subjects is very sketchy, so the reader who is interested in these areas is advised to refer to the corresponding literature. For instance, there are several books devoted completely or partially to the spectral theory of periodic operators [94, 341, 372, 389], so I do not go into the details of this deep theory, and present only some basic results, sometimes without complete proofs.

The numeration of theorems, formulas, etc. reflects the numbers of the chapter, the section and the item inside the section. For instance, the first theorem in section 1.1 will be referred to as theorem 1 inside the section, and as theorem 1.1.1 from other sections.

I am indebted to many people whose help enabled me to write this book. I would like to mention, first of all, the role of three people: Professor S. Krein, whose guidance and help has played a very important role in my life, and in this research in particular; Professor L. Zelenko, who made the suggestion to start this research, and with whom we obtained first results; Professor V. Palamodov, whose results and ideas were on many occasions my major sources of information and inspiration. I am thankful to Professors C. Berenstein, G. DaPrato, N. Firsova, B. Gramsch, S. Helgason, W. Kaballo, Yu. Korobeinik, E. Landis, J. Leiterer, V. Lin, A. Lunardi, F. Mantlik, A. Miloslavskii, A. Pankov, V. Papanicolaou, I. Pinchover, L. Ronkin, M. Shubin, B.A. Taylor, M. Zaidenberg, and V. Zhikov for valuable discussions and information. I am indebted to Professors A. Elcrat, I. Gohberg, K. Lancaster, K. Miller, L. Mogilevskaya, P. Parker and V. Papanicolaou and to my graduate student S. Lissianoi for their helpful comments on parts of the manuscript. I would like to thank Mrs. P. Altum, Mrs. C. MacDonald, Dr. L. Mogilevskaya and Ms. A. Olson for typing the text. I appreciate the help and understanding that I have had from my family. The work on some results of sections 1.5, 5.2 and 5.3 was partially supported by the NSF Grant No. DMS-9102111, and I would like to thank the National Science Foundation for this support.