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Well-Posedness of Parabolic Difference Equations

**A. Ashyralyev
P.E. Sobolevskii**

Translated from the Russian by A. Iacob

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Authors

A. Ashyralyev
Department of Mathematical Analysis
The Turkmen State University
31, Suparmyrat Turkmenbushy Shayoly
744000 Ashgabat
Turkmenistan

P.E. Sobolevskii
Institute of Mathematics
The Hebrew University
Givat Ram
91904 Jerusalem
Israel

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PREFACE

A well-known and widely applied method of approximating the solutions of problems of mathematical physics is the method of difference schemes. The main characteristics of difference schemes are their accuracy and stability. Modern computers allow the implementation of highly accurate difference schemes. Hence, a task of current interest is the construction and investigation of highly accurate difference schemes for various types of boundary value problems of mathematical physics. The present monograph is devoted to the construction of highly accurate difference schemes for parabolic boundary value problems, based on Padé approximation.

The convergence properties of these schemes and the estimates of their rate of convergence are determined by the type of stability they enjoy. The first stability estimates for such difference schemes were established in Hilbert space norms, which in applications led to L_2 estimates of stability and rate of convergence. However, in applications uniform estimates of the rate of convergence are far more interesting. Earlier such estimates were established on the basis of a maximum principle for the simplest difference schemes of low order of accuracy, and only for second-order equations. In the present monograph the study of the stability of difference schemes for parabolic equations is based on a new notion of positivity of a linear operator in a Banach space, which in applications allows one to deal with difference schemes of arbitrary order of accuracy and establish their stability and convergence in Hölder norms. Our investigations are based on the results of a new theory of interpolation of linear operators.

The type of stability that is the most interesting for application purposes is the so-called coercive stability. Establishing coercivity inequalities for solutions of difference schemes is important in applications because such inequalities allow one to obtain sharp, i.e., two-sided estimates of the rate of convergence. Such

inequalities for highly accurate difference schemes in Banach space norms can be also proved based on the theory of interpolation of linear operators.

We note that the existence of coercivity inequalities is equivalent to the natural notion of well-posedness of difference problems.

Let us give a brief account of the contents of our monograph. It consists of four chapters.

In Chapter 1 we study the well-posedness of the abstract Cauchy problem

$$v'(t) + Av(t) = f(t), \quad 0 \leq t \leq 1, \quad v(0) = v_0,$$

in various spaces $F(E)$ of functions defined on $[0, 1]$ with values in some Banach spaces. The chapter consist of five sections.

In Section 1 the Cauchy problem is studied in the space $C(E)$. It is shown that a necessary condition for well-posedness is that the semigroup $\exp\{-tA\}$ ($t \geq 0$) be analytic in E . In the general case this condition is not sufficient for well-posedness in $C(E)$.

The analyticity of the semigroup $\exp\{-tA\}$ is sufficient for well-posedness in $C(E)$ of the homogeneous Cauchy problem ($f \equiv 0$). Moreover, the solution of this problem enjoys additional smoothness for $t > 0$, and consequently the homogeneous Cauchy problem is well-posed in the weighted Hölder space $C_0^\alpha(E)$, which is smaller than $C(E)$. As it turns out, the general Cauchy problem, too, is well posed in $C_0^\alpha(E)$. This is discussed in Section 2.

In Section 3 we study the well-posedness of the Cauchy problem in the spaces $L_p(E)$, $p \in (1, \infty)$. A necessary condition for well-posedness is again the analyticity of the semigroup $\exp\{-tA\}$. This condition is also necessary for any $p \in (1, \infty)$ provided the Cauchy problem is well posed in $L_{p_0}(E)$ for some $p_0 \in (1, \infty)$. For example, if E is a Hilbert space one can take $p_0 = 2$.

A criterion for well-posedness of the Cauchy problem in $C(E)$ and $C_0^\alpha(E)$ is that the initial datum v_0 belong to the domain $D(A)$ of the operator A . In the case of the space $L_p(E)$ the element v_0 must belong to the so-called space of traces $E_{1-\frac{1}{p}}$. It turns out that the solution in $L_p(E)$ of the general Cauchy problem is a continuous function with values in $E_{1-\frac{1}{p}}$.

In Section 4 we isolate a rich family of spaces $E_{\alpha,q}$, $0 < \alpha < 1$, $1 \leq q \leq \infty$, for which one has continuous and dense embeddings $D(A) \subset E_{\alpha,q} \subset E$, such that the Cauchy problem is well posed in $L_q(E_{\alpha,q})$. From this and the results of

Section 3 we derive the well-posedness of this problem in the spaces $L_p(E_{\alpha,q})$ with $1 < p, q < \infty$.

Section 5 is devoted to the study of the well-posedness of the Cauchy problem in spaces of smooth functions $C_0^{\beta,\gamma}(E)$, $0 \leq \gamma \leq \beta$. A particular case of such spaces, obtained for $\gamma = \beta = \alpha$, appears earlier in Section 2. Although the spaces $C_0^{\beta,\gamma}(E)$ depend on two parameters, it turns out that the estimate in the corresponding coercivity inequality depends only on β . Consequently, one can choose γ arbitrarily in $[0, \beta]$. In particular, the Cauchy problem is well posed in the ordinary Hölder space $C^\alpha(E) = C_0^{\alpha,0}(E)$.

The study of difference Cauchy problems is initiated in Chapter 2. Therein we study the well-posedness of the difference Cauchy problem generated by an implicit difference scheme of first order of accuracy. Well-posedness of difference problems is understood as the existence of a coercive estimate that is uniform in the step τ of the difference grid. We succeed in establishing a sharp coercive estimate, i.e., a two-sided estimate of the solution error. Chapter 2 consists of six sections.

In Section 0 we give formulas and estimates for resolvents, and we establish the stability of the difference scheme in various spaces.

In Section 1 we study the well-posedness of the difference problem in the space $\mathcal{C}(E)$. From the well-posedness of the difference problem in $\mathcal{C}(E)$ follows the well-posedness of the differential Cauchy problem in $C(E)$, and hence the analyticity of the semigroup $\exp\{-tA\}$, $t \geq 0$. Since in the general case the differential problem is not well posed in $C(E)$, the difference problem cannot be stable uniformly in τ . This means that the coercive norm of the solution tends to infinity when $\tau \rightarrow 0+$. The study of the difference problem itself allows us to determine the order of growth of this norm to infinity.

In Section 2 we study the well-posedness of the difference problem in $C_0^\alpha(E)$. Here we generalize the results of Section 2 of Chapter 1. The proof is based on smoothness estimates of powers of the resolvent of the operator $-A$, which are equivalent to estimates of the semigroup $\exp\{-tA\}$, $t \geq 0$.

Section 3 is devoted to the well-posedness of the difference problem in the space $\mathcal{L}_p(E)$. As in the case of the differential problem (Section 3 of Chapter 1), the proof relies on the Benedeck-Calderón-Panzone extrapolation theorem. Here we use again the equivalence of the norms in the spaces of traces generated by the semigroup and the powers of the resolvent.

In Section 4 we study the well-posedness of the difference problem in the space $\mathcal{L}_p(E_{\alpha,q})$ and generalize the results of Section 4, Chapter 1. These results are based on the Cauchy-Riesz formula for functions of a strongly positive operator and on the introduction of an equivalent norm in the spaces $E_{\alpha,q}$ by means of the resolvent of this operator.

Finally, in Section 5 we are concerned with the well-posedness of the difference problem in difference analogues of spaces of smooth functions, and we generalize the results of Section 5, Chapter 1. The study relies on smoothness estimates of powers of the resolvent of a strongly positive operator, which are equivalent to the smoothness estimates for the semigroup $\exp\{-tA\}$, $t \geq 0$, given in Section 5 of Chapter 1.

Chapter 3 is devoted to the study of the well-posedness of the difference Cauchy problem generated by the Padé fractions $R_{j,l}(z)$ ($l-2 \leq j \leq l$) approximating the exponential function. It consists of six sections.

In Section 0 we study the stability of the difference problem. The investigation of the stability and well-posedness of the Padé difference schemes relies on the properties of the rational functions $R_{j,l}(z)$ that generate them. Here we give formulas and establish estimates for $R_{j,l}(z)$. Then we construct Padé difference schemes and give criteria for their stability in the $\mathcal{C}(E)$ -, $\mathcal{C}_0^\alpha(E)$ -, and $\mathcal{L}_p(E)$ -norms.

In Section 1 we study the well-posedness of the difference problem in the space $\mathcal{C}(E)$. Based on the estimates for $R_{j,l}(z)$ given in Section 0, we establish estimates for powers of the operator step $R_{j,l}(\tau A)$ ($l-2 \leq j \leq l$) that are analogous to the estimates of powers of the resolvent. Let us note that the estimates of powers of $R_{l,l}(\tau A)$ have “worse” properties compared with those of the operator steps $R_{l-2,l}(\tau A)$ and $R_{l-1,l}(\tau A)$. The estimates obtained enable us to establish stability and almost coercivity inequalities for Padé difference schemes in $\mathcal{C}(E)$.

Section 2 is devoted to the well-posedness of the difference problem in the space $\mathcal{C}_0^\alpha(E)$. Based on the estimates of Section 1 we derive smoothness estimates for powers of the operator step $R_{j,l}(\tau A)$. The estimates obtained allow us to establish stability and coercivity estimates in $\mathcal{C}_0^\alpha(E)$ for difference schemes with $j = l-2$, $l-1$ or even $j = l$. For odd $j = l$ such inequalities are established in a space $\tilde{\mathcal{C}}_0^\alpha(E)$, smaller than $\mathcal{C}_0^\alpha(E)$.

In Section 3 we study the well-posedness of the difference problem in the space $\mathcal{L}_p(E)$. This study relies on the fact that the norm of the powers of the operator step $R_{j,l}(\tau A)$ decays exponentially. The latter holds for $j = l-2$, $l-1$ in the case

in which the operator A is strongly positive with spectral angle $\phi(A, E) < \pi/(2l)$.

In Section 4 we study the well-posedness of the difference problem in the spaces $\mathcal{L}_p(E_{\alpha,q})$. We establish stability and coercivity estimates for the Padé difference schemes for any j, l such that $l - 2 \leq j \leq l$ in $\mathcal{L}_q(E_{\alpha,q})$, $1 \leq q \leq \infty$, in the case of an arbitrary strongly positive operator A , and for $j = l - 2$, $l - 1$ in $\mathcal{L}_p(E_{\alpha,q})$, $1 < p, q < \infty$, in the case of a strongly positive operator A with spectral angle $\phi(A, E) < \pi/(2l)$.

In Section 5 we study the well-posedness of the Cauchy problem in spaces of smooth functions. Here we generalize the results of Sections 2 and 4. The stability and coercivity estimates are now of two types. The first covers the case of Padé difference schemes with $j = l - 2$, $l - 2$ or $j = l$ an even number, while the second covers the case of Padé difference schemes with $j = l$ an odd number.

In Chapter 4 we obtain stability and coercivity estimates for highly accurate difference schemes for the approximate solution of the Cauchy problem for parabolic equations. The chapter consists of three sections.

In Section 1 we deal with elliptic difference operators with constant coefficients. We construct a difference operator A_h that approximates a given elliptic operator A of arbitrary order with constant coefficients in $\mathbf{R}^n$. For the operator A_h we study the Green function (fundamental solution) of the resolvent equation. We establish point estimates of the difference derivatives of the Green function, which go over into the corresponding estimates for the differential case when the grid step tends to zero. We should point out that here a new form of estimates for derivatives of the difference Green function for large absolute values of the spectral parameter is exhibited.

In Section 2 we study fractional spaces in the case of an elliptic difference operator. We prove the uniform-in- h positivity of the operator A_h in $L_p = L_p(\mathbf{R}_h^n)$, $1 \leq p \leq \infty$. We show that the $E'_{\alpha,p}(L_{ph}, A_h)$ -norms are equivalent, uniformly in h , to the difference norms $W_p^{m\alpha}(\mathbf{R}_h^n)$ for $0 < \alpha < 1/m$. Finally, we prove the coercivity of the elliptic difference problem in the spaces $W_p^{m\alpha}(\mathbf{R}_h^n)$.

In Section 3 we show how the results about difference schemes obtained in Chapter 3 for abstract equations and the results of Sections 1 and 2 of Chapter 4 yield stability and coercivity estimates for parabolic difference equations.

The results presented in this monograph are due to the authors, and also to our colleagues A. E. Polichka, Kh. A. Alibekov, Hoàng Văn Lai, and Yu. A. Smirnitskiĭ. We gathered here only part of our investigations on the theory of difference

equations, begun in 1967 in Voronezh, at the Department of Functional Analysis and Operator Equations of Voronezh University. A brief survey of all these investigations can be found in the “Comments on the Literature” that conclude the monograph.