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Differentiable Operators and Nonlinear Equations

**Victor Khatskevich
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Translated from the Russian by Mircea Martin

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Table of Contents

Introduction

Chapter 0: Preliminaries

1. Sets and relations	1
2. Topological spaces	5
3. Convergence. Directedness	8
4. Metric spaces	13
5. Spaces of mappings	16
6. Linear topological spaces	19
7. Normed spaces	21
8. Linear operators and functionals	24
9. Conjugate space. Conjugate operator	27
10. Weak topology and reflexivity	29
11. Hilbert spaces	31

Chapter I: Differential calculus in normed spaces

1. The derivate and the differential of a nonlinear operator	35
2. Lagrange formula and Lipschitz condition	39
3. Examples of Fréchet differentiable operators	41
4. Lemmas about differentiable operators	43
5. Partial derivatives	45
6. Multilinear operators. Duality. Homogeneous forms	48
7. Higher order derivatives	50
8. Complete continuity of operators and of their derivatives	55

VI

Chapter II: Integration in normed spaces

1. Riemann - Stieltjes integrals of vector-functions 61
2. Pettis integral and the connection with Riemann - Stieltjes integral 65
3. Antiderivatives of vector-functions. Integral representations 66
4. Integrals of operators in Banach spaces 71

Chapter III: Holomorphic (analytic) operators and vector-functions on complex Banach spaces

1. Differentiability in complex and real sense.
Cauchy - Riemann conditions 76
2. The ρ -topology and holomorphy 80
3. Cauchy integral theorems and their consequences 85
4. Uniqueness theorems and maximum principles
5. Schwartz Lemma and its generalizations 92
6. Uniformly bounded families of ρ -holomorphic (holomorphic) operators.
Montel property 98

Chapter IV: Linear operators

1. The spectrum and the resolvent of a linear operator 103
2. Spectral radius 108
3. Resolvent and spectrum of the adjoint operator 111
4. The spectrum of a completely continuous operator 114
5. Normally solvable operators 117
6. Noether and Fredholm operators 119
7. Projections. Splittable operators 122
8. Invariant subspaces 127

Chapter V: Nonlinear equations with differentiable operators

1. Fixed points. Banach principle 133
2. Non-expansive operators 137
3. Fixed points for differentiable operators 143
4. Some applications of fixed point principle 147
5. Implicite and inverse operators. Connection with fixed points 160

Chapter VI: Nonlinear equations with holomorphic operators

1. s-fixed points for holomorphic operators.
A converse of Banach principle 171
2. Criteria for the existence of an s-fixed point and
its extension with respect to a parameter 177
3. Regular fixed points. Geometric criteria 182

4. Apriori estimates and the extension of an s-solution to the boundary of the domain.....	189
5. Local inversion of holomorphic operators and a posteriori error estimates.....	195
6. Single-valued small solutions in some degenerate cases.....	200
Chapter VII: Banach manifolds	
1. Basic definitions.....	211
2. Smooth mappings.....	213
3. Submanifolds.....	214
4. Complex manifolds and Stein manifolds.....	218
Chapter VIII: Non-regular solutions of nonlinear equations	
1. Ramification of solutions. Statement of the problem.....	223
2. Equations of ramification.....	225
3. Equations of ramification for an analytic operator. The problem of the coefficients.....	231
4. The description of the set of fixed points for an analytic operator.....	232
Chapter IX: Operators on spaces with indefinite metric	
1. Spaces with indefinite metric.....	239
2. Angle operators.....	242
3. Plus-operators.....	244
4. Symmetric properties of a plus-operator and its adjoint.....	249
5. The problem of invariant semi-definite subspaces.....	258
6. An application of fixed point principles for holomorphic operators to the invariant semi-definite subspace problem.....	262
References.....	267
List of Symbols.....	277
Subject Index.....	279

Introduction

We have considered writing the present book for a long time, since the lack of a sufficiently complete textbook about complex analysis in infinite dimensional spaces was apparent. There are, however, some separate topics on this subject covered in the mathematical literature. For instance, the elementary theory of holomorphic vector-functions and mappings on Banach spaces is presented in the monographs of E. Hille and R. Phillips [1] and L. Schwartz [1], whereas some results on Banach algebras of holomorphic functions and holomorphic operator-functions are discussed in the books of W. Rudin [1] and T. Kato [1].

Apparently, the need to study holomorphic mappings in infinite dimensional spaces arose for the first time in connection with the development of nonlinear analysis. A systematic study of integral equations with an analytic nonlinear part was started at the end of the 19th and the beginning of the 20th centuries by A. Liapunov, E. Schmidt, A. Nekrasov and others. Their research work was directed towards the theory of nonlinear waves and used mainly the undetermined coefficients and the majorant power series methods. The most complete presentation of these methods comes from N. Nazarov.

In the forties and fifties the interest in Liapunov's and Schmidt's analytic methods diminished temporarily due to the appearance of variational calculus methods (M. Golomb, A. Hammerstein and others) and also to the rapid development of the mapping degree theory (J. Leray, J. Schauder, G. Birkhoff, O. Kellog and others). These new methods were particularly attractive since they enabled the study of many classes of nonlinear equations, and therefore they were highly developed. (Important results were obtained by M. Krasnoselski, P. Zabreiko, V. Odinetz, Yu. Borisovich and B. Sadovskii.) However, these new techniques retarded the development of spe-

cific methods for solving equations with an analytic nonlinear part. That is why in the sixties some mathematicians (P. Rybin, V. Pokornyi, M. Vainberg, V. Trenogyn and others) interested in the theory of integral equations and their applications returned to the Liapunov-Schmidt and Nekrasov-Nazarov analytic methods.

At the same time the theory of functions of one or several complex variables was enriched with more significant and subtle results. Parallel with these achievements, the first results on holomorphic mappings on infinite dimensional spaces appeared in the works of A. Cartan, R. Phillips, L. Nachbin, L.Harris, T. Suffridge, W.Rudin, M. Herve, E. Vesentini, J.-P. Vigue, P. Mazet, K. Goebel, and of many others.

We consider that it is now about the right time “to set a bridge” between nonlinear analysis and the theory of holomorphic mappings on infinite dimensional spaces. Of course, to this end it is necessary to put together results and techniques from the homology theory, sheaf theory, vector fields theory and from a lot of other modern theories in analysis — a task difficult to achieve within the limits of but one book. That is why we decided to start this vast project, by presenting only the theory of differentiable and holomorphic mappings on Banach spaces, as well as some prerequisites from functional analysis and topology.

In all chapters with the exception of Chapter 0 which has the character of a dictionary, we tried to give a complete account of definitions and proofs, and to make this book accessible not only to specialists, but also to students and to those engineers who are currently using the solutions of some specific integral and differentiable equations.

We conclude the work by mentioning the interesting relationship between the theory of holomorphic mappings and the theory of linear operators on spaces with indefinite metrics. More precisely, our last chapter is a brief exposition of the theory of spaces with indefinite metrics and of some relevant applications of the holomorphic mappings theory in this setting.

In closing, we draw our readers to a few technical points. Throughout the book we strove to use a uniform notation for objects of the same type. The most used notations are presented in Chapter 0. At the end of the book we give a list of some standard symbols, and also a subject index. We used the symbols “◁” and “▷” for the beginning and the end of a proof, respectively. The references in the text contain the name(s) of the author(s) followed by a number in brackets, which corresponds to the one in the reference list.