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Michael Struwe

Leonid Polterovich

**The Geometry of the
Group of Symplectic
Diffeomorphism**

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Author's address:

School of Mathematical Sciences
Tel Aviv University
69978 Tel Aviv
Israel

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Preface

The group of Hamiltonian diffeomorphisms $\text{Ham}(M, \Omega)$ of a symplectic manifold (M, Ω) plays a fundamental role both in geometry and classical mechanics. For a geometer, at least under some assumptions on the manifold M , this is just the connected component of the identity in the group of all symplectic diffeomorphisms. From the viewpoint of mechanics, $\text{Ham}(M, \Omega)$ is the group of all admissible motions. What is the minimal amount of energy required in order to generate a given Hamiltonian diffeomorphism f ? An attempt to formalize and answer this natural question has led H. Hofer [H1] (1990) to a remarkable discovery. It turns out that the solution of this variational problem can be interpreted as a *geometric quantity*, namely as the distance between f and the identity transformation. Moreover this distance is associated to a canonical biinvariant metric on $\text{Ham}(M, \Omega)$. Since Hofer's work this new geometry has been intensively studied in the framework of modern symplectic topology. In the present book I will describe some of these developments.

Hofer's geometry enables us to study various notions and problems which come from the familiar finite dimensional geometry in the context of the group of Hamiltonian diffeomorphisms. They turn out to be very different from the usual circle of problems considered in symplectic topology and thus extend significantly our vision of the symplectic world. Is the diameter of $\text{Ham}(M, \Omega)$ finite or infinite? What are the minimal geodesics? How can one find the length spectrum? In general, these questions are still open. However some partial answers do exist and will be discussed below.

There is one more, to my taste even more important reason why it is useful to have a canonical geometry on the group of Hamiltonian diffeomorphisms. Consider a time dependent vector field $\xi_t, t \in \mathbb{R}$ on a manifold M . The ordinary differential equation

$$\dot{x} = \xi(x, t)$$

on M defines a flow $f_t : M \rightarrow M$ which takes any initial condition $x(0)$ to the solution $x(t)$ at time t . The trajectories of the flow form a complicated system of curves on the manifold. Usually, in order to understand the dynamics, one should travel along the manifold and thoroughly study behaviour of

the trajectories in different regions. Let us change the point of view and note that our flow can be interpreted as a simple geometric object – a single curve $t \rightarrow f_t$ on the group of all diffeomorphisms of the manifold. One may hope that geometric properties of this curve reflect the dynamics, and thus complicated dynamical phenomena can be studied by purely geometric tools. Of course, the price we pay is that the ambient space – the group of diffeomorphisms – is infinite-dimensional. Moreover, we face a serious difficulty: in general there are no canonical tools to perform geometric measurements on this group. Remarkably, Hofer's metric provides such a tool for systems of classical mechanics. In Chapters 8 and 11 we will see some situations where this ideology turns out to be successful.

As it often happens with very young and fast developing areas of mathematics, the proofs of various elegant statements in Hofer's geometry are technical and complicated. Thus I selected the simplest non-trivial versions of the main phenomena (according to my taste, of course) without making an attempt to present them in the most general form. For the same reason many technicalities are omitted. Though formally speaking this book does not require a special background in symplectic topology (at least necessary definitions and formulations are given), the reader is cordially invited to consult two remarkable introductory texts [HZ] and [MS]. Both of them contain chapters on the geometry of the group of Hamiltonian diffeomorphisms. I have tried to minimize the overlaps. The book contains a number of exercises which presumably will help the reader to get into the subject.

This book arose from two sources. The first one is lectures given to graduate students, namely mini-courses at Universities of Freiburg and Warwick, and a *Nachdiplom*-course at ETH, Zürich. The second source is the expository article [P8] which contains a brief outline of the material presented below.

Let me introduce briefly the main characters of the book. A diffeomorphism f of a symplectic manifold (M, Ω) is called *Hamiltonian* if it can be included into a compactly supported Hamiltonian flow $\{f_t\}$ with $f_0 = \mathbb{1}$ and $f_1 = f$. Such a flow is defined by a Hamiltonian function $F : M \times [0; 1] \rightarrow \mathbb{R}$. In the language of classical mechanics F is the energy of a mechanical motion described by $\{f_t\}$. We interpret the total energy of the flow as a length of the corresponding path of diffeomorphisms:

$$\text{length}\{f_t\} = \int_0^1 \max_{x \in M} F(x, t) - \min_{x \in M} F(x, t) dt.$$

Define a function

$$\rho : \text{Ham}(M, \Omega) \times \text{Ham}(M, \Omega) \rightarrow \mathbb{R}$$

by

$$\rho(\phi, \psi) = \inf \text{length}\{f_t\},$$

where the infimum is taken over all Hamiltonian flows $\{f_t\}$ which generate the Hamiltonian diffeomorphism $f = \phi\psi^{-1}$. It is easy to see that ρ is non-negative, symmetric, vanishes on the diagonal and satisfies the triangle inequality. Moreover, ρ is biinvariant with respect to the group structure of $\text{Ham}(M, \Omega)$. In other words, ρ is a biinvariant pseudo-distance. It is a deep fact that ρ is a genuine distance function, that is $\rho(\phi, \psi)$ is strictly positive for $\phi \neq \psi$. The metric ρ is called Hofer's metric.

The group $\text{Ham}(M, \Omega)$ and Hofer's pseudo-distance ρ are introduced in Chapters 1 and 2 respectively. In Chapter 3 we prove that ρ is a genuine metric in the case when M is the standard symplectic linear space $\mathbb{R}^{2n}$. Our approach is based on Gromov's theory of holomorphic discs with Lagrangian boundary conditions which is presented in Chapter 4.

Afterwards we turn to the study of basic geometric invariants of $\text{Ham}(M, \Omega)$. There is a (still open!) conjecture that the diameter of $\text{Ham}(M, \Omega)$ is infinite. In Chapters 5–7 we prove this conjecture for closed surfaces.

In Chapter 8 we discuss the concept of the growth of a one-parameter subgroup $\{f_t\}$ of $\text{Ham}(M, \Omega)$ which reflects the asymptotic behaviour of the function $\rho(\mathbb{1}, f_t)$ when $t \rightarrow \infty$. We present a link between the growth and the dynamics of $\{f_t\}$ in the context of invariant tori of classical mechanics.

In many interesting situations the space $\text{Ham}(M, \Omega)$ has complicated topology, and in particular a non-trivial fundamental group. For an element $\gamma \in \pi_1(\text{Ham}(M, \Omega))$ set $\nu(\gamma) = \inf \text{length}\{f_t\}$, where the infimum is taken over all loops $\{f_t\}$ of Hamiltonian diffeomorphisms (that is periodic Hamiltonian flows) which represent γ . The set

$$\{\nu(\gamma) \mid \gamma \in \pi_1(\text{Ham}(M, \Omega))\}$$

is called the length spectrum of $\text{Ham}(M, \Omega)$. In Chapter 9 we present an approach to the length spectrum which is based on the theory of symplectic fibrations. An important ingredient of this approach is Gromov's theory of pseudo-holomorphic curves which is discussed in Chapter 10. In Chapter 11 we give an application of our results on the length spectrum to the classical ergodic theory.

In Chapters 12 and 13 we develop two different approaches to the theory of geodesics on $\text{Ham}(M, \Omega)$. One of them is elementary, while the other requires a powerful machinery of Floer homology. Chapter 13 suggests to the reader a brief visit to Floer homology.

Finally, in Chapter 14 we deal with non-Hamiltonian symplectic diffeomorphisms, which appear naturally in Hofer's geometry as isometries of $\text{Ham}(M, \Omega)$. In addition, we formulate and discuss the famous flux conjecture which states that the group $\text{Ham}(M, \Omega)$ is closed in the group of all symplectic diffeomorphisms endowed with the C^∞ -topology.

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