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Projective Geometry and Formal Geometry

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For Mihaela, Miruna and Ciprian

Preface

The aim of this book is to introduce the reader to some modern methods of projective geometry involving, in one way or another, certain techniques of formal geometry. Some of these methods are illustrated in the first part through the proofs of a number of results of projective geometry (of a rather classical flavor and having an obvious interest in themselves). In three of these proofs the crucial idea is to consider the first infinitesimal neighbourhood of a certain closed subvariety Y of an ambient projective variety X . In other words, the schemes with nilpotents can play a very important role, sometimes even when one deals with classical problems of projective geometry. When we want to study a certain closed embedding $Y \subset X$ of algebraic varieties, quite often we are naturally led to consider all higher order infinitesimal neighbourhoods of Y in X . In other words, one is led to work with the formal completion X/Y of X along Y , which is a formal scheme, introduced into algebraic geometry by Zariski and Grothendieck in the 1950s. The formal scheme X/Y is very well suited to study the given embedding $Y \subset X$, and is an analogue of the concept of tubular neighbourhood of a submanifold of a complex manifold.

The second part starts with a systematic and rather self-contained presentation of the basic concepts of formal geometry that are necessary to prove a number of basic results concerning the extension of formal-rational functions to rational functions. As we mentioned before, this theory is due to Zariski and Grothendieck. It was subsequently developed by Hironaka, Matsumura, Hartshorne, Chow, Faltings and others. It provides powerful tools for proving deep results in algebraic geometry, such as the famous Zariski connectivity theorem. In fact, the second part will make clear the deep relationship between formal geometry and connectivity results in projective geometry (such as the Fulton–Hansen connectivity theorem). The formal geometry is closely related with a number of questions in complex analysis and complex-analytic geometry (considered by Severi, Remmert, Van de Ven, Barth, Rossi, Chow and others), such as the extension problem of meromorphic functions or (more generally) of certain analytic entities. This relationship is also explained in the second part of the book.

Besides, we have tried to apply several results of formal geometry (especially concerning the extension of formal-rational functions to rational functions) to projective geometry, in particular to the geometry of homogeneous spaces. These aspects are also closely related with various kinds of Grothendieck–Lefschetz theorems in projective geometry. We hope to convince the reader that the interplay between projective and formal geometry is very important and fruitful.

The book is self-contained modulo the basic concepts and results of algebraic geometry and cohomology (for instance Hartshorne's textbook [73] is in principle enough as background material). It contains many recent results which have not appeared in any other textbook. The second part could be considered in particular as an introduction to formal schemes. The book addresses persons working in algebraic geometry and in complex geometry, as well as all mathematicians working in other fields, but interested in questions related to the ones touched upon here.

Acknowledgment

Part 1 is based on a Ph.D. course I gave at the University of Milan (Italy) in the Spring 1999. I am grateful to Antonio Lanteri for inviting me at the University of Milan to give this "corso di dottorato INDAM", to Leonede De Michele for supporting this idea, and to the Istituto Nazionale di Alta Matematica (Roma) for financial support. I enjoyed the pleasant and stimulating atmosphere of the whole group of algebraic geometers of Milan: Alberto Alzati, Marina Bertolini, Elisabetta Colombo, Antonio Lanteri, Marino Palleschi, Cristina Turrini, and Alfonso Tortora. I thank them all. My thanks also go to Francesco Russo and his family for their friendship and warm hospitality I enjoyed during my stay in Milan (Rho).

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Introduction

This book is devoted to some aspects concerning the interplay between projective geometry and formal geometry. It consists of two parts. The main aim of Part 1 is to introduce the reader to some modern aspects and methods of projective geometry. The material presented here has been conceived to touch three important themes: the connection between projective geometry and deformations of quasi-homogeneous singularities, cohomological dimension and connectedness results, and applications of formal geometry to projective geometry. Part 1 is based on a series of lectures the author gave in the Spring of 1999 at the Dipartimento di Matematica, Università degli Studi di Milano (see [13]).

As far as the first theme is concerned, Chapter 1 starts with the classical problem of classifying the extensions in $\mathbb{P}^{n+1}$ of a given closed subvariety Y in $\mathbb{P}^n$. This problem was very popular among Italian algebraic geometers in the first part of the last century. A renewed interest in it started in the 1970s. As a consequence of the modern methods used, the classical picture has been completed and better understood and many new results have been proved. In particular, the whole story has been put in a new perspective. It is beyond our scope to discuss all its aspects and results here. We rather restrict ourselves to presenting a remarkable result due to Zak–L’vovsky (Theorem 1.2), also because its proof presented here is relevant for the philosophy of this book. This proof relies on another fundamental result due to Mori–Sumihiro–Wahl (Theorem 1.3) which is also of interest in itself. Chapters 1–3 provide the proofs of these two results and a number of applications, comments and examples. The condition involved in the Zak–L’vovsky theorem (the surjectivity of the Zak map) is better understood in the case of curves in terms of the so-called Gaussian maps (see Chapter 4). The study of Gaussian maps was initiated in 1987 by J. Wahl with the main motivation of understanding the geometry of curves lying on $K3$ -surfaces. In Chapter 5 we present a result of Schlessinger which relates the deformation theory of the vertex of the affine cone C_Y over a smooth projectively normal closed subvariety Y in $\mathbb{P}^n$ with the projective geometry of Y . In particular, it becomes transparent that the surjectivity of the Zak map is naturally interpreted in terms of deformations of the vertex of the affine cone C_Y over $Y \subset \mathbb{P}^n$.

The second theme is presented in Chapter 7. First we prove a special case of a result of Hartshorne–Lichtenbaum (conjectured by Grothendieck) which asserts that the cohomological dimension of a quasi-projective variety U of dimension n is $\leq n - 1$ if and only if U is not a projective variety. This result is then applied to

prove a generalization of the Fulton–Hansen connectedness theorem to weighted projective spaces. Then some applications of this connectedness result are given. We will come back to Fulton–Hansen’s result in the second part of the book to reinterpret it in the framework of formal functions and then to prove a significant improvement of it.

In Chapters 6 and 8 we present two results whose proofs involve, in an essential way, the first infinitesimal neighbourhood of a closed subvariety $X \subset \mathbb{P}^n$. The first result, due to Van de Ven, characterizes the linear subspaces as the only irreducible smooth closed subvarieties of $\mathbb{P}^n$ for which the normal sequence splits. The second, due to Ellingsrud–Gruson–Peskine–Strømme, gives a necessary and sufficient condition for a curve Y lying on a complete intersection surface X in $\mathbb{P}^n$ to be the scheme-theoretic intersection of X with a hypersurface of $\mathbb{P}^n$ (see Theorem 8.4). The methods of proving this latter result also yield a geometric proof of a result of Barth–Larsen which asserts that $\text{Pic}(X) = \mathbb{Z}$ for every smooth closed subvariety X of $\mathbb{P}^n$ of dimension $\geq \frac{n+2}{2}$. We hope that the method of using the first infinitesimal neighbourhood will convince the reader less familiar with formal methods that formal geometry deserves to be studied and applied to projective geometry; in particular, this provides an excellent motivation for the second part of the book. At the end of Chapter 8, two appendices are added. In Appendix A we derive a special case of Lefschetz’s theorem for the Picard group from the well-known topological Lefschetz theorem on hyperplane sections (see e.g. [105]). In Appendix B we review some elementary facts about cyclic covers.

A very preliminary version of Part 2 is our survey [14]. The main sources of Part 2 are Grothendieck–Dieudonné [66], Hironaka–Matsumura [81], Hartshorne [75], Faltings [47], [48], Gieseker [58], Bădescu [11], [15], Bădescu–Schneider [17], [18] and Bădescu–Beltrametti–Ionescu [19]. Fix an irreducible projective variety X over an algebraically closed field k . Let $K(X)$ be the field of rational functions of X , and let $K(X/Y)$ denote the ring of formal-rational functions of X along a closed subscheme Y of X . Sometimes we shall call the formal-rational functions simply formal functions. If X is normal and Y is connected of positive dimension, then $K(X/Y)$ is a field. The field $K(X)$ can be considered in a natural way as a subfield of $K(X/Y)$, and the extension problem of formal-rational functions asks when can any formal-rational function on X along Y be extended to a rational function on X , i.e. when do we have $K(X) = K(X/Y)$? If the latter equality holds, according with the terminology of Hironaka and Matsumura [81], we say that Y is G_3 in X .

The formal functions have been introduced and studied systematically in algebraic geometry by Zariski in [151]. In particular, Zariski used them to prove his famous connectedness theorem. Later on this theory was considerably extended and deepened by Grothendieck in the framework of formal schemes, by providing powerful tools (such as Grothendieck’s existence theorem, see [66] III, Theorem (5.1.4)) to study related questions (see [67], [68]).

There is an analogous problem in the complex-analytic setting. Given a pair (X, Y) consisting of a complex irreducible projective variety X and a closed subvariety Y of X , let U be a connected open neighbourhood of Y in X (in the complex topology), and denote by $\mathcal{M}(U)$ the $\mathbb{C}$ -algebra of all meromorphic functions on U . We get the field extensions $K(X) \subseteq \mathcal{M}(U) \subseteq K(X/Y)$, and therefore the extension problem for meromorphic functions can be solved if it can be solved for formal functions, i.e. the equality $K(X) = \mathcal{M}(U)$ is a consequence of the equality $K(X) = K(X/Y)$.

Extension of analytic or formal objects like functions or subvarieties is a rather classical subject. The earliest reference we are aware of is a paper of Severi [137], in which he proved that any meromorphic function defined in a small complex connected neighbourhood of a smooth hypersurface in $\mathbb{P}^n(\mathbb{C})$ ($n \geq 2$) can be extended to a meromorphic function on $\mathbb{P}^n(\mathbb{C})$. After some partial extension of Severi's result by Remmert and Van de Ven [123], Barth [21] generalized the above-mentioned result of Severi to arbitrary closed positive-dimensional connected subvarieties of $\mathbb{P}^n(\mathbb{C})$. On the other hand, Hironaka [78] extended Severi's result to arbitrary smooth projective varieties X and smooth hypersurfaces Y in X with ample normal bundle. His approach is algebraic and he actually considers the extension problem of formal-rational functions along Y . Further important contributions to the extension problem of meromorphic functions or analytic objects are due to Griffiths [63], Chow [35], Rossi [124], and others.

The second part (which, as we said before, is a natural continuation of part one) contains six chapters. In Chapter 9 we present the basic definitions and concepts as well as some of the fundamental results (due to Grothendieck, Hironaka, Hironaka–Matsumura, Hartshorne, Faltings, and others) concerning the problem of extending formal-rational functions to rational functions. This (rather long) chapter can also be considered as an introduction to formal schemes and formal functions.

Chapter 10 starts by explaining the relationship between formal geometry and the so-called Grothendieck–Lefschetz conditions $\text{Lef}(X, Y)$ and $\text{Lef}^*(X, Y)$ for a pair consisting of a closed subvariety Y of a projective ambient variety X . These conditions are crucial to prove algebro-geometrically various types of Lefschetz theorems (see [68]). Moreover it turns out that, under some additional mild hypotheses, one has

$$\text{Lef}^*(X, Y) \implies Y \text{ G3 in } X \implies \text{Lef}(X, Y).$$

In Chapter 10 one also relates, via Serre's GAGA [135], the problem of extending formal-rational functions with the analogous problem in Complex Analysis of extending meromorphic functions defined in a small complex neighbourhood of a closed subvariety.

Motivated by the strong connection of formal geometry with connectedness theorems in algebraic geometry given by Theorem 9.21 and Corollary 9.22, in Chapter 11 we interpret the Fulton–Hansen connectedness theorem in terms of

formal-rational functions. Then we strengthen this result to Theorem 11.2. One of the relevant consequences of this strengthening asserts that the diagonal embedding Δ_Y of the product $Y \times Y$ is $G3$ in $Y \times Y$ for every closed irreducible subvariety Y of $\mathbb{P}^n$ such that $\dim(Y) > \frac{n}{2}$ and $n \geq 3$. In other words, for every closed irreducible subvariety Y as above, the diagonal embedding $\Delta_Y \subset Y \times Y$ enjoys an important positivity property. Some further applications of Theorem 11.2 are given. We also include Zak's original proof of the finiteness of the Gauss map which used formal functions. At the end of this chapter we prove an impossibility cohomological criterion for a smooth projective variety Y of dimension ≥ 2 to occur as the zero locus of a global section of an ample vector bundle on a smooth projective variety X with prescribed normal bundle (Theorem 11.25).

In Chapter 12 one proves two criteria for extending formal-rational functions. The first one (Theorem 12.6) is a refinement of a result of Hartshorne (see [75], [74], or also Theorem 9.28). This result will be applied in the next chapter to homogeneous spaces. The second result (Theorem 12.9) asserts that the $G3$ condition holds when Y is an effective Cartier divisor on X such that $X \setminus Y$ is a semi-affine variety in the sense of Goodman–Landman [61]. In Appendix C following Chapter 12 one proves a result of Hartshorne concerning the ampleness of the normal bundle of a curve lying in an abelian variety and an ampleness criterion of Gieseker for vector bundles on curves.

Chapter 13 deals with formal functions on homogeneous spaces and presents some fundamental results due to Chow, Faltings, and others. In particular, applying the main result of the previous chapter, we get more elementary proofs of some results of Faltings [47] concerning formal functions on homogeneous spaces. Various applications are given, e.g. concerning the s -geometrically non-degenerate closed subvarieties of an abelian variety. The concept of s -geometrically non-degenerate subvariety of an abelian variety is due to Ran–Debarre [122], [42], and some of its main properties are presented in Appendix D following Debarre [42].

In Chapter 14 the projective manifolds carrying quasi-lines are studied (following [19]). A quasi-line Y of a smooth projective complex variety X of dimension $n \geq 2$ is a smooth rational curve $Y \subset X$ whose normal bundle is isomorphic to the normal bundle of a line in $\mathbb{P}^n$. One proves (among other things) that the quasi-lines of X are precisely those smooth curves $Y \subset X$ with ample normal bundle for which the canonical restriction map $\text{Pic}(X) \rightarrow \text{Pic}(Y(1))$ has finite cokernel (see Theorem 14.2). One shows by examples that there are pairs (X, Y) , (X', Y') , with Y quasi-line in X and Y' quasi-line in X' (with X and X' projective manifolds of dimension $n \geq 3$), such that (X, Y) and (X', Y') are not formally equivalent (i.e. $X_{/Y} \not\cong X'_{/Y'}$). The projective manifolds carrying quasi-lines are special cases of rationally connected manifolds (see [94], or also [93]). Some of their properties are also included (e.g. they are stable under small projective deformations).

Throughout this book the standard terminology and notation of algebraic geometry (see e.g. [73] or [138] and [139]) are used, unless otherwise specified.