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Jorge Buescu

Exotic Attractors

From Liapunov Stability to Riddled Basins

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The case for my life, then, or for that of anyone else who has been a mathematician in the same sense in which I have been one, is this: that I have added something to knowledge, and helped others to add more; and that these somethings have a value which differs in degree only, and not in kind, from that of the creations of the great mathematicians, or of any of the other artists, great or small, who have left some kind of memorial behind them.

G. H. Hardy, *A Mathematician's Apology.*

To both of you.

Contents

Preface

This book grew out of the work developed at the University of Warwick, under the supervision of Ian Stewart, which formed the core of my Ph.D. Thesis. Most of the results described were obtained in joint work with Ian; as usual under these circumstances, many have been published in research journals over the last two years. Part of Chapter 3 was also joint work with Peter Ashwin. I would like to stress that these were true collaborations. We worked together at all stages; it is meaningless to try to identify which idea originated from whom.

While preparing this book, however, I felt that a mere description of the results would not be fitting. First of all, a book is aimed at a wider audience than papers in research journals. More importantly, the work should assume as little as possible, and it should be brought to a form which is pleasurable, not painful, to read.

These are the reasons why I decided to include historical remarks at the end of each chapter. In order to make the book self-contained, a thorough review of the essential concepts is needed from the very beginning. Why not complement the presentation of the concepts as we know them today with an overview of the way they originated and evolved?

A cautionary note, though. The above comments should not mislead the reader into thinking that this book is intended to be a systematic development or a thorough review of specific areas of Dynamics. Let me be clear about this: it is *not*. It is a research report on two different problems in Dynamics, the largest common divisor of which is the concept of attractor of a dynamical system. This fact emerges at the most apparent level of the book: its title. Thus "exotic attractors" are not a class of attractors with specific properties; the adjective "exotic" is not a mathematical characterization. It merely indicates that in both problems the attractors in question, not necessarily related to each other, have certain distinctive features which may or may not be a part of the intuitive mental picture of an attractor.

Every time some mathematical structure is added, care is taken to explore its consequences. However, concepts and their relationships are only developed up to the point of their usefulness to the problems at hand. This should not be taken to mean that further developments do not exist or are not useful.

I would like the book to reflect its genesis in research, describing the positive results and living topics but also the dead ends and open problems. I decided, besides reporting where we stand, not to delete the footprints left on the way. It is bad enough that Mathematics often presents to outsiders the image of a polished, perfect, finished body of absolute truths. We all know Mathematics isn't so: I didn't pretend it is when writing this book. Sometimes conciseness pays the price of this option, as the reader will undoubtfully appreciate in Chapter 2.

I don't know if these goals have been achieved in a balanced and articulate manner; only the reader can be the judge of that. I can only hope that he enjoys reading this work as much as I enjoyed writing it.

Chapter 1 consists of a thorough review of the concepts used in later chapters, assuming very little at the outset – in fact nothing beyond the definition of a (semi-)dynamical system. Several results are established on the nature of Liapunov and asymptotic stability and on the implications of transitivity (i.e. existence of a dense orbit) for the topology of the connected components of an invariant set. Some relations between the topological and ergodic properties of invariant sets and attractors are described. We then shortly discuss Axiom A, asymptotically stable and Milnor attractors. Counterexamples are given to illustrate the finer points of the results and definitions.

In Chapter 2 the first body of results is described. Suppose A is a Liapunov-stable transitive set with infinitely many connected components – a prototype of which is the Cantor set arising at the Feigenbaum limit of period-doubling. By the results in Chapter 1, the quotient by components generates a Cantor set K on which the induced map is transitive. The main Theorem states that this map is an adding machine. This result is shown to hold under the weaker assumption that A is a stable ω-limit set by the more elegant inverse limit constructions. These results, which in a sense are surprising since they lie close to the foundations of Dynamics, have non-trivial consequences. For interval maps they imply that every point in the adding machine is a limit point of periodic points whose periods are very naturally determined by the adding machine itself.

In Chapter 3 the following problem is addressed. Suppose a dynamical system possesses an invariant submanifold restricted to which it has an asymptotically stable attractor A. When is A an attractor for the full system, and in what sense? This question is particularly meaningful when A is chaotic, and has arisen under different guises (e.g. synchronization problems or systems with symmetries) in applications-oriented literature over the last decade. The dynamics near A for the full system is characterized by the spectrum of normal Liapunov exponents. This spectrum determines the points where A: (1) ceases to be asymptotically stable, possibly developing a riddled basin; (2) ceases to be an attractor; (3) becomes a transversely repelling chaotic saddle. A sufficient condition for the creation of a riddled basin in transition (1) is provided. With these results an adequate bifurcation theory is developed. We study in some detail analytical and physical systems exhibiting this behaviour, including an experimental system of symmetrically coupled electronic oscillators.

My greatest debt is of course to Ian Stewart. Ian didn't offer me a fish: he taught me how to do my own fishing. He was always there when I needed him but also gave me the freedom to pursue my own ideas – and make my own mistakes. Needless to say, if any have survived up to this stage (which I hope not) only I am to blame.

The mathematical atmosphere at Warwick was always stimulating. I have benefited enormously from regular discussions with many Warwick mathematicians, notably Greg King, Robert MacKay, Mario Micallef, David Mond, Mark Pollicott, David Rand and especially Peter Walters. I also had important discussions for the brewing of these ideas with Ian Melbourne. It was a great pleasure to work with Peter Ashwin; many fine points only became clear after challenging but fruitful brainstormings.

Many people helped me in many ways during these years. My Mother gave me constant and unfailing support. I have met wonderful people at Warwick, without whom my work would have been much less bearable. Their friendship often brought brightness to the darker hours; I thank them all now that my work has seen daylight. I would like to thank especially Inês Cruz and Carlos Monteiro for their friendship, Claudio Arezzo for the long discussions of Mathematics and Pat and Lito de Baubeta for everything.

I also thank the publishers at Birkhäuser for showing a keen interest in my work and for their encouragement – which includes their patience at the final stage of preparation of the book!

The research leading to this work was conducted while on leave from the Departamento de Matemática of Instituto Superior Técnico. During this period I had partial support from JNICT-CIENCIA through grant BD/1073/90-RM and from Fundação Calouste Gulbenkian.

The work leading to this book was partially supported by projects JNICT-PBIC/P/MAT/2140/95 and JNICT–PRAXIS/2/2.1/MAT/199/94.

My years at Warwick also witnessed the happiest moments in my life: the blossoming of a family. The time has come to thank Catarina and Henrique for all their sacrifices.

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