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Global Aspects of Classical Integrable Systems

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Foreword

This book gives a complete global geometric description of the motion of the two dimensional harmonic oscillator, the Kepler problem, the Euler top, the spherical pendulum and the Lagrange top. These classical integrable Hamiltonian systems one sees treated in almost every physics book on classical mechanics. So why is this book necessary? The answer is that the standard treatments are not complete. For instance in physics books one cannot see the monodromy in the spherical pendulum from its explicit solution in terms of elliptic functions nor can one read off from the explicit solution the fact that a tennis racket makes a near half twist when it is tossed so as to spin nearly about its intermediate axis. Modern mathematics books on mechanics do not use the symplectic geometric tools they develop to treat the qualitative features of these problems either. One reason for this is that their basic tool for removing symmetries of Hamiltonian systems, called regular reduction, is not general enough to handle removal of the symmetries which occur in the spherical pendulum or in the Lagrange top. For these symmetries one needs singular reduction. Another reason is that the obstructions to making local action angle coordinates global such as monodromy were not known when these works were written.

The point of view adopted in this book is to start with a somewhat unfamiliar abstract mathematical model of the physical system such as the study of the geodesic flow of a left invariant metric on the three dimensional rotation group. Using the symplectic geometric formulation of Hamiltonian mechanics we then show that the equations of motion agree with those found by more traditional methods for a well known physical system, namely, the force free rigid body or Euler top. This justifies our mathematical model. We do not try to build our model from fundamental physical principles. We have not written a book on mechanics or Hamiltonian particle dynamics. We only discuss five special integrable systems, which is a very small sample of the rich variety of general Hamiltonian systems. Moreover the behavior of the solutions of these integrable systems is much more regular than the nearly unpredictable motion of a general Hamiltonian system such as the three body problem.

Our main goal is to understand the global geometric features of our model integrable systems. The main tool we use is reduction to remove the symmetries and to obtain a system with one degree of freedom. This allows us to determine the range and the topology of every fiber of the energy momentum mapping of the system. The topology of a fiber corresponding to a singular value of the energy momentum mapping is of great interest. Physically, these motions are simpler than the general motion and therefore are easier to study experimentally. Mathematically, these fibers contain a relative equilibrium of the system, that is, a motion which is also an orbit of the symmetry group. For instance, in the spherical pendulum the relative equilibria are circular orbits on the 2-sphere which lie in a plane parallel to and below the equator. Other examples are the regular precession and sleeping motions of the Lagrange top. Finally, to complete the qualitative picture, we describe how the fibers of the energy momentum map fit together. Sometimes this involves showing that the monodromy of certain torus bundles are nontrivial. That this phenomenon happens in the spherical pendulum and the Lagrange top was not known until the 1980s.

This book is written from a bottom up approach with examples being given prominence over theory. The examples are treated in a uniform way. First the mathematical model is described and then the equations of motion are derived. Next the symmetries and corresponding integrals are obtained and it is shown that the given problem is Liouville integrable. Finally, the geometry of the level sets of the energy momentum map, which gives a complete geometric description of the motion, are obtained by first using reduction to remove the symmetries and then reconstructing the geometry from the geometry of the reduced system. This program may seem to be excessively lengthy. There are two reasons why we have followed it. First, our procedure gives complete answers, whereas short cut ones do not. Second, in carrying out our program the reader sees enough detail in the text to be able to understand the arguments without having to look at the theory. The theory given in chapters VI through XI is what the authors feel is the minimum necessary to justify all the unproved assertions in the examples.

This book was not written to be read in a sequential fashion. We strongly encourage the reader to browse.

Introduction

The mathematical pendulum

We begin by looking at the mathematical pendulum.

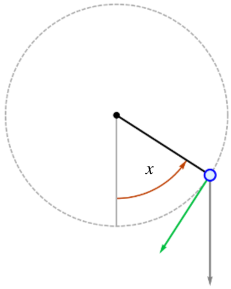

Figure 0.1. The mathematical pendulum.

Let $T^*\mathbf{R}$ be the *cotangent bundle* of $\mathbf{R}$, which we identify with $\mathbf{R}^2$ and give coordinates (x,y). The canonical symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}y$ on $T^*\mathbf{R}$ is the element of oriented area on $\mathbf{R}^2$. Consider the *Hamiltonian system* $(H, T^*\mathbf{R}, \omega)$ with *Hamiltonian*

$$H : T^*\mathbf{R} \to \mathbf{R} : (x,y) \mapsto \tfrac{1}{2}y^2 - \cos x.$$

▷ The following argument shows that the *Hamiltonian vector field* X_H on $T^*\mathbf{R}$ corresponding to the Hamiltonian H is

$$X_H(x,y) = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} = y\frac{\partial}{\partial x} - \sin x \frac{\partial}{\partial y}. \tag{1}$$

(0.1) **Proof**: By definition of Hamiltonian vector field, see appendix A §3,

$$\mathrm{d}H(p)z_p = \omega(p)(X_H(p), z_p) \tag{2}$$

for every $z_p = (v_p, w_p)$ in the tangent space $T_p(T^*\mathbf{R})$ to $T^*\mathbf{R}$ at p. Let $X_H(p) = \big(X(p), Y(p)\big)$. Now $\mathrm{d}H(p)z_p = \frac{\partial H}{\partial x}v_p + \frac{\partial H}{\partial y}w_p$. Moreover, $\omega(p)(X_H(p), z_p)$ is the oriented area spanned by the parallelogram with sides $X_H(p)$ and z_p, that is,

$$\omega(p)(X_H(p), z_p) = \det\begin{pmatrix} X(p) & v_p \\ Y(p) & w_p \end{pmatrix} = X(p)w_p - Y(p)v_p.$$

Therefore (2) is equivalent to

$$\frac{\partial H}{\partial x}v_p + \frac{\partial H}{\partial y}w_p = -Y(p)v_p + X(p)w_p \tag{3}$$

for every $(v_p, w_p) \in \mathbf{R}^2$. In (3) choose $(v_p, w_p) = (1,0)$. Then $X(p) = \frac{\partial H}{\partial y} = y$. Next choose $(v_p, w_p) = (0,1)$. Then $Y(p) = -\frac{\partial H}{\partial x} = -\sin x$. □

Note that (1) may be written as the second order differential equation

$$\ddot{x} = -\frac{\mathrm{d}}{\mathrm{d}x}(-\cos x) = -\sin x. \tag{4}$$

By Newton's second law of motion, an integral curve of (1) describes the motion of a particle of unit mass under a force coming from the *potential* $V : \mathbf{R} \to \mathbf{R} : x \mapsto -\cos x$.

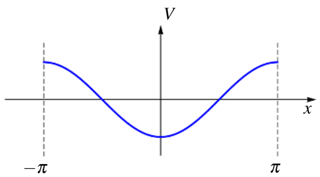

Figure 0.2. The graph of the potential $V(x) = -\cos x$.

Thus H is the total energy of the particle, namely the sum of the *kinetic* and potential ▷ energy. We now show that H is a Morse function on $T^*\mathbf{R}$.

(0.2) **Proof**: The point $p = (x,y)$ is a critical point of H if and only if $X_H(p) = 0$, that is, if and only if

$$0 = \frac{\partial H}{\partial y} = \sin x \qquad \text{and} \qquad 0 = \frac{\partial H}{\partial x} = y.$$

Thus $\{p = (n\pi, 0) \in \mathbf{R}^2 | n \in \mathbf{Z}\}$ is the set of *critical points* of H. The corresponding *critical value* of H is -1 if n is even or 1 if n is odd. Since the Hessian of H at p is

$$D^2H(p) = \left.\begin{pmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial y} \\ \frac{\partial^2 H}{\partial y \partial x} & \frac{\partial^2 H}{\partial y^2} \end{pmatrix}\right|_p = \begin{pmatrix} \cos n\pi & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (-1)^n & 0 \\ 0 & 1 \end{pmatrix},$$

H is a Morse function, because $D^2H(p)$ is nondegenerate. □

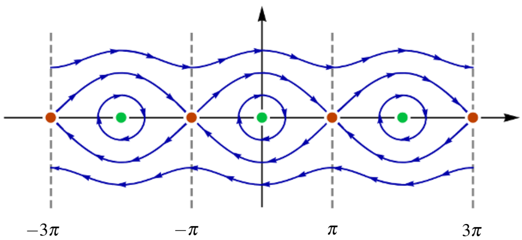

Figure 0.3. The level sets of $H(x,y) = \frac{1}{2}y^2 - \cos x$.

When $n = 2k$ the Morse index of $D^2H(n\pi, 0)$ is zero, and so the critical points $(2k\pi, 0)$ are relative minima of H; whereas when $n = 2k+1$ the *Morse index* of $D^2H(n\pi, 0)$ is one, and so the critical points $(2k\pi, 0)$ are saddle points of H, see figure .3. Using the Morse lemma, see appendix F §1, there is a neighborhood U_k of $(2k\pi, 0)$ in the open strip $((2k-1)\pi, (2k+1)\pi) \times \mathbf{R}$ such that for h slightly greater than -1 the *level set* $H^{-1}(h) \cap U_k$ is diffeomorphic to a circle. Since H has no critical values in the interval $(-1, 1)$, by the Morse isotopy lemma, see appendix F §3, we deduce that for every $h \in (-1, 1)$ the level set $H^{-1}(h) \cap U_k$ is diffeomorphic to a circle. Thus for $h \in (-1, 1)$ the whole level set $H^{-1}(h)$ is diffeomorphic to a countable disjoint union of circles. If $h \geq 1$, then $H^{-1}(h)$ is the union of the graphs of two smooth functions $y_{\pm} = \pm\sqrt{2(h + \cos x)}$. The graphs of $y_{\pm}$ are disjoint if $h > 1$. On the other hand, if $h = 1$, then the graphs of $y_{\pm} = \pm 2\cos\frac{1}{2}x$ intersect only at the points $((2k+1)\pi, 0)$. There they intersect transversely as can be seen by applying the Morse lemma at the points $((2k+1)\pi, 0)$. Thus we have obtained a picture of the level curves of H as given in figure .3.

To simplify the topology of the level sets of H, we make use of the fact that H is invariant under the *translation symmetry*

$$\mathbf{Z} \times T^*\mathbf{R} \to T^*\mathbf{R} : \big(n, (x, y)\big) \mapsto (x + 2n\pi, y). \tag{5}$$

Thus H induces a function $\widetilde{H}$ on the space of orbits $T^*\mathbf{R}/2\pi\mathbf{Z}$. Concretely, this orbit space is identified with the cotangent bundle T^*S^1 of the circle S^1. Here S^1 is

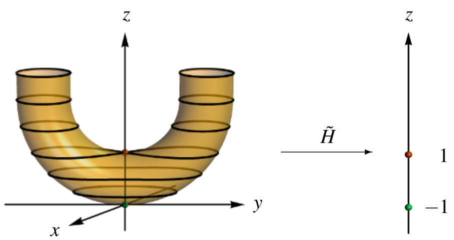

Figure 0.4. The graph of $\widetilde{H}(x, y) = \frac{1}{2}y^2 - \cos x$ with $(x, y) \in T^*S^1$.

thought of as the orbit space $\mathbf{R}/2\pi\mathbf{Z}$ of the real numbers modulo 2π. Geometrically, T^*S^1 is the cylinder $S^1 \times \mathbf{R}$ which is obtained from $\mathbf{R}^2$ by cutting along the vertical lines $x = 0$ and $x = 2\pi$ and then pasting the edges together. Applying this process to figure .3 gives figure .4 which depicts the level sets of $\widetilde{H}$ and hence the orbits of the induced Hamiltonian vector field $X_{\widetilde{H}}$. A short argument using Newton's second law shows that the second order differential equation

$$\ddot{x} = -\sin x \qquad x \bmod 2\pi$$

describes the motion of a particle of mass one on the unit circle under the influence of a constant vertical downward unit force, see figure .1.

From figure .4 we see that the topological circle, defined by the component of the level set $\widetilde{H}^{-1}(h)$ $(h > 1)$ lying in the upper half cylinder, is very different from the topological circle defined by the level set $\widetilde{H}^{-1}(h)$ $(-1 < h < 1)$. The first circle is not contractible in TS^1 to a point whereas the second circle is. Hence it is impossible to continuously deform the first circle into the second one. This difference in the topological disposition of the two circles corresponds to the physical fact that for small energy the particle oscillates about the bottom of the circle, while for large energy the particle loops over the top of the circle.

Exercises

1. Let (x,y) be canonical coordinates on $T^*\mathbf{R} = \mathbf{R}^2$ with symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}y$. Suppose that the Hamiltonian $H : T^*\mathbf{R} \to \mathbf{R}$ is a sum of kinetic and potential energy, that is, $H(x,y) = \frac{1}{2}y^2 + V(x)$, where $V : \mathbf{R} \to \mathbf{R}$.

   a) Find a potential function V such that the zero level set of H is connected, compact and has one singular point which is a cusp.

   b) Construct a polynomial Hamiltonian on $T^*\mathbf{R}$ whose zero level set is an n-leaf clover.

   c) Show that there is *no* smooth Hamiltonian which is the sum of kinetic and potential energy which has a 3-leaf clover as a level set.

   d) For smooth V with countable many isolated critical points give a topological characterization of the critical level sets of H.

2. Construct a Hamiltonian function on S^2 which is a Morse function with two critical points. Draw its level sets. Construct a vector field on S^2 with only one equilibrium point and sketch its orbits. Show that this vector field is not Hamiltonian.

3. a) On $\mathbf{R}^2$ consider the action $\cdot$ of $\mathbf{Z}^2$ defined by $(n,m)\cdot(x,y) = (x+n, y+m)$. The orbit space $\mathbf{R}^2/\mathbf{Z}^2$ is a two dimensional torus T^2, which may be modeled by a square with the opposite sides identified. The symplectic form $\Omega = \mathrm{d}x \wedge \mathrm{d}y$ on $\mathbf{R}^2$ induces a symplectic form $\widetilde{\Omega}$ on T^2. Show that the vector field $\widetilde{X}$ on T^2 induced by the Hamiltonian vector field $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ on $\mathbf{R}^2$ is *not* Hamiltonian on $(T^2, \widetilde{\Omega})$.

   b) Sketch the orbits of a Hamiltonian vector field on $(T^2, \widetilde{\Omega})$ where the Hamiltonian is a Morse function with the fewest number of critical points.

   c) Construct a vector field X on T^2 with *two* equilibrium points.

   d)* Show that a smooth function on T^2 must have at least three critical points. Find a smooth function on T^2 with exactly three critical points. Sketch its level sets.

   e) Deduce that the vector field X constructed in c) is not Hamiltonian.

4. Let M be a compact connected orientable smooth two dimensional manifold with volume form Ω. In what follows we show that every integral curve of a Hamiltonian vector field X_H of a one degree of freedom Hamiltonian system (H, M, Ω) is either an equilibrium point, a periodic orbit, or is asymptotic to an equilibrium point as $t \to \pm\infty$. For $m \in M$ let $\gamma : \mathbf{R} \to M : t \mapsto \varphi_t^H(m)$ be the integral curve of X_H through m. The *ω-limit set* $\omega(\gamma)$of γ is the closure of the set $\bigcap_{T>0} \{\varphi_t^H(m) \big| t \geq T\}$.

   a) Show that $\omega(\gamma)$ is nonempty.

b) If γ is an equilibrium point or a periodic orbit of X_H, show that $\omega(\gamma) = \gamma$. Is the converse true?

c)* If γ is not a periodic orbit of X_H, show that $\omega(\gamma)$ is a critical point of H, that is, an equilibrium point of X_H.

4. (Period energy relation for the mathematical pendulum.) When $|h| < 1$ show that the period of an integral curve of the mathematical pendulum which starts at $(x_+, 0)$ where $0 < x_+ = x_+(h) < \pi$ and $h + \cos x_+ = 0$, is given by

$$\tau(h) = 2 \int_{-x_+}^{x_+} \frac{1}{\sqrt{2(h + \cos x)}}\, dx.$$

Show that $\tau = 4K(\sqrt{(h+1)/2})$, where K is the complete elliptic integral of the first kind, see the exercises of chapter 1. Deduce that

a) $\tau(-1) = 2\pi$, $\tau(1) = \infty$ and $\tau'(-1) = \pi/4$.

b) τ is a real analytic function on $(-1, 1)$.

c)* $\tau' > 0$ on $(-1, 1)$. Hint: show that τ satisfies a differential equation.

5. a) Suppose that a particle moves on the graph of $y = f(x)$ under the influence of gravity and that the origin is a stable equilibrium point. Determine the shape of the graph of f so that the period of oscillation of the particle about the origin is a constant independent of the energy.

b)* Show that a) is equivalent to the fact the derivative of the area enclosed by a level set of the Hamiltonian of the particle with respect to the Hamiltonian itself is a constant. Hint: see appendix D §1.

6. (Reduction of discrete symmetry of mathematical pendulum.)

a) (Discrete symmetry.) Show that

$$\zeta : S^1 \times \mathbf{R} \to S^1 \times \mathbf{R} : (x, y) \mapsto (-x, -y). \tag{6}$$

generates a $\mathbf{Z}_2$-*symmetry* of the mathematical pendulum. Show that the fundamental domain $\mathscr{D}$ of the $\mathbf{Z}_2$-action on T^*S^1 generated by ζ is the piece of the cylinder in figure .4, which lies in the half space $y \geq 0$ with the points $(\pm x, 0)$ on the circle $\partial\mathscr{D} = T^*S^1 \cap \{y = 0\}$ identified. Deduce that the orbit space $P = T^*S^1/\mathbf{Z}_2$ is homeomorphic to a cone on S^1 with vertex at the $\mathbf{Z}_2$-orbit corresponding to the point $(0,0) \in T^*S^1$.

b) Show that the algebra of real analytic invariant functions of the abelian group $\mathbf{Z}_2$ generated by ζ is generated by

$$\tau_1 = \cos x, \qquad \tau_2 = y \sin x, \qquad \tau_3 = \tfrac{1}{2}y^2 - \cos x \tag{7}$$

subject to the relation

$$C(\tau) = \tfrac{1}{2}\tau_2^2 - (\tau_3 + \tau_1)(1 - \tau_1^2) = 0, \quad |\tau_1| \leq 1 \;\&\; \tau_3 \geq -1, \tag{8}$$

which defines the orbit space P. Draw a picture of the semialgebraic variety P.

c) (Reduced Poisson bracket.) In order to have dynamics on the $\mathbf{Z}_2$-reduced space P we first need a Poisson bracket $\{\,,\,\}_{\mathbf{R}^3}$ on $C^\infty(\mathbf{R}^3)$. A calculation shows that

$$\begin{aligned}
\{\tau_1,\tau_2\} &= \tau_1^2 - 1 = \frac{\partial C}{\partial \tau_3}\\
\{\tau_2,\tau_3\} &= 2\tau_1(\tau_3+\tau_1) + \tau_1^2 - 1 = \frac{\partial C}{\partial \tau_1}\\
\{\tau_3,\tau_1\} &= \tau_2 = \frac{\partial C}{\partial \tau_2}
\end{aligned}$$

Then for every $F,\, G \in C^\infty(\mathbf{R}^3)$ we have $\{F,G\} = \sum_{i,j} \frac{\partial F}{\partial \tau_i}\frac{\partial G}{\partial \tau_j}\{\tau_i,\tau_j\}$. We say that a function f on P is smooth if there is a smooth function F on $\mathbf{R}^3$ such that $f = F|P$. Let $C^\infty(P)$ be the space of smooth functions on P. Then $(P, C^\infty(P))$ is a *differential space*, which is *subcartesian* because P is a semialgebraic variety. On $C^\infty(P)$ we define a *Poisson bracket* $\{\,,\,\}_P$ as follows. Suppose that $f,g \in C^\infty(P)$. Then there are $F,G \in C^\infty(\mathbf{R}^3)$ such that $f = F|P$ and $g = G|P$. Let $\{f,g\}_P = \{F,G\}_{\mathbf{R}^3}|P$. Because the defining function C (8) of the orbit space P is a *Casimir* in the Poisson algebra $\mathscr{A} = (C^\infty(\mathbf{R}^3), \{\,,\,\}_{\mathbf{R}^3}, \cdot)$, the collection $\mathscr{I}$ of all smooth functions on $\mathbf{R}^3$, which vanish identically on P, is a *Poisson ideal* in $\mathscr{A}$. Consequently, the Poisson bracket $\{\,,\,\}_P$ is well defined. So $\mathscr{B} = \big(C^\infty(P) = C^\infty(\mathbf{R}^3/\mathscr{I}), \{\,,\,\}_P, \cdot\big)$ is a Poisson algebra.

d) (Reduced dynamics.) Consider the derivation $-\mathrm{ad}_H$ on the Poisson algebra $\mathscr{A}$. This derivation gives rise to the $\mathbf{Z}_2$-reduced Hamiltonian vector field X_H on the subcartesian differential space $(P, C^\infty(P))$ associated to the $\mathbf{Z}_2$*-reduced Hamiltonian* $H : P \to \mathbf{R} : \tau \mapsto \tau_3$. On $\mathbf{R}^3$ the integral curves of $-\mathrm{ad}_H$ satisfy

$$\begin{aligned}
\dot{\tau}_1 &= \{\tau_1, H\}_P = \{\tau_1,\tau_3\}_P = -\tau_2\\
\dot{\tau}_2 &= \{\tau_2, H\}_P = \{\tau_2,\tau_3\}_P = 2\tau_1(\tau_3+\tau_1) + \tau_1^2 - 1\\
\dot{\tau}_3 &= \{\tau_1, H\}_P = \{\tau_1,\tau_3\}_P = 0.
\end{aligned}$$

The equality $\dot{\tau}_3 = 0$ shows that H is an integral of X_H. Check that C (8) is also an integral of X_H. A calculation shows that $-\mathrm{ad}_H$ leaves $C^{-1}(0)$, $\{\tau_3 + \tau_1 = 0\}$, and $\{\tau_1 = \pm 1\}$ invariant. Thus the *reduced space* P is invariant under the flow of $-\mathrm{ad}_H$. Consequently, the *reduced Hamiltonian vector field* X_H on P is $-\mathrm{ad}_H|P$. Because the Hamiltonian vector field $X_{\widetilde{H}}$ of the mathematical pendulum is complete, the reduced vector field X_H is complete. Its flow φ_t^H is a 1-parameter group of diffeomorphisms of P. In fact, for $p \in H^{-1}(e)$ the closure of the integral curve $\{\varphi_t^H(p) \in P \,\big|\, t \in \mathbf{R}\}$ is a connected component of the level set $H^{-1}(e)$, since a level set of the reduced Hamiltonian H is compact.