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Bertrand Duplantier • Vincent Rivasseau
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Henri Poincaré, 1912–2012

Poincaré Seminar 2012

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Foreword

This book is the thirteenth in a series of Proceedings for the *Séminaire Poincaré*, which is directed towards a broad audience of physicists, mathematicians, and philosophers of science.

The goal of this Seminar is to provide up-to-date information about general topics of great interest in physics. Both the theoretical and experimental aspects of the topic are covered, generally with some historical background. Inspired by the *Nicolas Bourbaki Seminar* in mathematics, hence nicknamed *"Bourbaphy"*, the Poincaré Seminar is held twice a year at the Institut Henri Poincaré in Paris, with written contributions prepared in advance. Particular care is devoted to the pedagogical nature of the presentations, so that they may be accessible to a large audience of scientists.

This new volume of the Poincaré Seminar Series, *Poincaré*, 1912–2012, corresponds to the sixteenth such seminar, held on November 24, 2012, on the occasion of the centennial of the death of HENRI POINCARÉ in 1912. Its aim was to offer in four lectures a scholarly approach to Poincaré's unfathomable genius and creativity in mathematical physics and mathematics. They covered his work on electromagnetism, optics, and relativity, on the three-body problem and the foundations of chaos theory, the slow but irreversible integration of probability theory with Poincaré's mathematical tools, and, last but not least, the proof of the famous Uniformization Theorem of Riemann surfaces in its six successive versions. A movie, which presented the week-long exchanges among six eminent scientists about the "harmony and chaos" in Poincaré's legacy, was projected in front of a delighted audience.

There were, in late nineteenth-century physics, a few problems that exceeded the mathematical power and the conceptual ingenuity of contemporary physicists, important instances of which were optical diffraction, the nature of the ether, and the electrodynamics of moving bodies. Poincaré, though not strictly a physicist, greatly contributed to the solution of these problems and to the new mathematical physics that emerged at the dawn of the next century. At the same time, he developed a new philosophy of physics, that has inspired many other philosophers to this day. These great achievements keep challenging historians. In the first contribution to this volume, entitled "Poincaré's Light", OLIVIER DARRIGOL, a leading historian of science, uses light as a guiding thread through much of Poincaré's physics and philosophy, and thus explains the originality and fertility of his approaches.

This thread began with the Sorbonne lectures on the mathematical theories of light, in which the young Poincaré applied his superior mathematical skills to the theory of diffraction, and in which at the same time he developed a structuralist-conventionalist philosophy of physical theory. His subsequent reflections on the foundations of electromagnetism largely depended on this philosophy and on the relation between optics and electromagnetism. He understood that the conceptual difficulties of this theory had to do with the fundamental role of light in the measurement of space and time, thus anticipating Einstein's operational critique of time and simultaneity. His analysis of the optical ether led Poincaré to the principle of relativity in the late 1890s, and to the fully covariant formulation of Lorentz's theory in 1905. Yet, as shown in this learned article, he ended up preserving the ether as the reference for true space and time and as the carrier of all momentum and energy, as a consequence of the same philosophy that had permitted his earlier questioning of the naive mechanical ether.

The second article, the authoritative "Poincaré and the Three-Body Problem" by Alain Chenciner, offers an exquisitely detailed perspective on the monumental work of Poincaré on this subject. It was only incidentally that Poincaré became a mathematical physicist and an astronomer, first holding for a decade the Sorbonne Chair of Calculus of Probability and Mathematical Physics. But it was also a stroke of luck for science. With the author, we follow the steps of Poincaré in the 1889 Memoir, *Sur le problème des trois corps et les équations de la dynamique* (and its corrected version of 1890 in *Acta Mathematica*), which contained the first mathematical description of chaotic behavior in a dynamical system, won him the prize awarded at the occasion of the 60th birthday of King Oscar II of Sweden, and brought him international fame. Then came the extraordinary three volumes of *Les méthodes nouvelles de la mécanique céleste*, published in 1892, 1893 and 1899, which revolutionized celestial mechanics and established the modern theory of dynamical systems. The later works, such as the Poincaré-Birkhoff theorem, appeared in 1912, the very year of Poincare's untimely death.

The author underlines the progression from the search of periodic and quasi-periodic solutions of the three-body problem to the advanced analytic, topological, geometric and probabilistic aspects of the *Méthodes nouvelles*. Highlights are a precise definition of the Poincaré-Lindstedt perturbative series at any order and the breakthrough discovery of their divergence, which led to the modern aspects of the theory (return map, Poincaré's recurrence theorem, ergodic behavior and chaos).

Chenciner takes the time to explain in great detail the mathematics used, and often invented, by a Poincaré inspired by the physical context of his subject, which makes this review extremely readable and valuable. The text is illustrated by beautifully hand-drawn figures and peppered with wide excerpts of citations, usually in French with English translation. They thus allow the reader to enjoy Poincaré's unique and vivid style, while entering into the depth of his foundational work in mechanics.

The thorough study by Laurent Mazliak, “Poincaré’s Odds”, offers an original and scholarly presentation of the work by Poincaré in probability theory. A challenge in dealing with this topic is that probability penetrated Poincaré’s mathematics almost against his intentions, forcing his hand several times. He had indeed received his scientific training in the second half of the 19th century, at a time when Newton’s mechanics and Laplace’s determinism were the Alpha and Omega of the scientific explanation of the physical world.

The first appearance of the word *probability* in Poincaré’s works, in the 1890 probabilistic statement of his famous theorem of recurrence for dynamical systems, involved no intrinsic randomness and was rather a convenient way of expressing the rarity of exceptional non-recurrent trajectories. However, following Boltzmann’s and Maxwell’s works, statistical mechanics was taking a growingly important role on the scientific stage. After several incisive exchanges with the British physicist Peter Guthrie Tait about Poincaré’s 1892 book on Thermodynamics, the latter decided to study and teach the kinetic theory of gases, and he became over the years convinced of the unavoidability of the new approach. During the last twenty years of his life, he devoted theoretical studies, such as the “method of arbitrary functions”, as well as philosophical works, such as the chapter on calculus of probabilities in *Science and Hypothesis*, to the question of randomness and its measurement, with a major aim: to identify the situations in which the scientific method required the use of probability theory. A careful examination by Mazliak of these texts, and especially of the two editions of Poincaré’s treatise on probability (1896 and 1912) provides us with a clear vision of the evolution of the latter’s thinking on the subject. Poincaré seemingly showed little taste for new mathematical techniques, such as measure theory and Lebesgue integration, which could have provided decisive tools to tackle numerous problems.

As a forerunner to its modern development, Poincaré was considered the leading French authority on probability theory. This explains his role during the Dreyfus Affair, when in 1906 he was asked to shed light on the protagonist Bertillon’s dubious use of probability concepts. Émile Borel, Poincaré’s main direct disciple, developed his approach to a great extent, and influenced the use of probabilities not only in physics, but also in numerous social sciences involving risk and decision. Poincaré’s philosophy of randomness also attracted the keen interest of several Czech academics, such as the philosopher Karel Vorovka and the mathematician Bohuslav Hostinský. Partly through them, Poincaré’s considerations on card shuffling and ergodicity met a considerable development in the years following 1920, with the emergence of the general theory of Markov chains, which became so fundamental to probability theory in the 20th century.

In “Henri Poincaré and the Uniformization of Riemann Surfaces”, François Béguin takes us on a fascinating journey following the maturation of a great mind confronted by an important problem. Through twenty-six years and six succesive versions, the author describes for us the unfolding and deepening of Poincaré’s thoughts on the subject up to his ultimate result, which combines a simple and definitive mathematical formulation with an elegant and physical proof. The author thus unveils how uniformization connects together the various facets of Poincaré's work and exemplifies his distinctive fusion of mathematics and physics.

When Poincaré started his work on uniformization, his aim was to understand multi-valued complex analytic functions, and in particular how to parametrize an algebraic curve, as defined by $f(x,y) = 0$, such that both complex variables x and y are single-valued functions of a single other variable z. To this effect, he introduced functions and their invariance groups that he called "Fuchsian". In a famous burst of creativity, he realized that the transformations of his "Fuchsian groups" were identical to those of non-Euclidean (hyperbolic) geometry. This led him to the first two uniformization theorems (1881), which applied first to the Riemann sphere minus a finite number of real points, and then to algebraic Riemann surfaces, modulo a finite number of points. These brilliant steps brought the young Poincaré in contact with Felix Klein for a fruitful scientific correspondence and rivalry, which resulted in their simultaneous statement in 1882 of the "true" uniformization theorem for algebraic Riemann surfaces by means of the so-called continuity method. This was followed in 1883 by the uniformization of (multi-valued) analytic functions, a general result which required the invention by Poincaré of the universal cover of a Riemann surface. The fifth approach of 1898 involved solving the Liouville equation as an alternative method for uniformizing algebraic Riemann surfaces.

Finally, in order to solve Hilbert's 22nd problem, namely to complete the uniformization program for non-algebraic Riemann surfaces, Koebe and Poincaré independently proved in 1907 the modern version of the theorem: *Every simply connected Riemann surface is biholomorphic to the Riemann sphere, the complex plane, or the unit disk.* Their methods were different, and Poincaré used a beautiful, physically inspired, "sweeping method" (i.e., the sweeping of electric charges out of a sequence of disks) which is described in full detail in this article, together with Koebe's simplified proof. Modern mathematical physics, from conformal field theory to string theory and quantum gravity rests on these solid foundations.

In the final chapter, "Harmony and Chaos, On the Figure of Henri Poincaré", Philippe Worms briefly describes the circumstances and aims of the film that bears the same title, and was projected on the day of the Seminar. Six renowned mathematicians and physicists spent several days of joyful reflection on Poincaré's work in an isolated country house. They speak of geometry, intuition and truth; with this film we embark on a poetical journey unveiling a new mathematical world. The filmmaker successfully focuses, through mathematical scenes and physical experiments, on their emotional relationship to an often elusive truth. It is evoked for us by Poincaré's first words in *La valeur de la science*:

> *"Seeking the truth should be the aim of our activity, it is the only worthy end.* [...] *If we want to gradually free men from material concerns, it is so they can use their newly reacquired freedom to study and gaze at the truth."*

At the end of the same text, we find "harmony" and "beauty":

> *"Mais ce que nous appelons la réalité objective, c'est, en dernière analyse, ce qui est commun à plusieurs êtres pensants, et pourrait être commun à tous ; cette partie commune, comme nous le verrons, ce ne peut être que l'harmonie exprimée par des lois mathématiques.*
>
> *C'est donc cette harmonie qui est la seule réalité objective, la seule vérité que nous puissions atteindre ; et si j'ajoute que l'harmonie universelle du monde est la source de toute beauté, on comprendra quel prix nous devons attacher aux lents et pénibles progrès qui nous la font peu à peu mieux connaître."*

This book, by the breadth of topics covered in Poincaré's legacy, should be of broad interest to mathematicians, physicists and philosophers of science. We further hope that the continued publication of this series of Proceedings will serve the scientific community, at both the professional and graduate levels. We thank the Commissariat à l'Énergie Atomique et aux Énergies Alternatives (Division des Sciences de la Matière), the Daniel Iagolnitzer Foundation, and the École polytechnique for sponsoring this Seminar. Special thanks are due to Chantal Delongeas for the preparation of the manuscript.

Saclay & Orsay
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