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Robert Roussarie

Bifurcations of Planar Vector Fields and Hilbert's Sixteenth Problem

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Contents

Preface	ix
1 Families of Two-dimensional Vector Fields	
1.1 Vector fields on surfaces of genus 0	1
1.1.1 The phase space	1
1.1.2 The Poincaré-Bendixson property	1
1.1.3 Phase portrait	3
1.2 A first approach to Bifurcation Theory	6
1.2.1 General definitions	7
1.2.2 Singularities of finite codimension. The saddle-node bifurcations	9
1.2.3 Bifurcations of singular points versus bifurcations of periodic orbits. The Bogdanov-Takens bifurcation	10
2 Limit Periodic Sets	
2.1 Organizing centers for bifurcations of limit cycles	18
2.1.1 Definition of limit periodic sets	18
2.1.2 The structure of limit periodic sets	20
2.2 The cyclicity property	22
2.2.1 Definition of cyclicity for limit periodic sets	22
2.2.2 The finite cyclicity conjecture. Local reduction of Hilbert's 16 th problem	23
2.2.3 A program for solving the existential Hilbert's problem	25
3 The 0-Parameter Case	
3.1 Blowing up of singularities of vector fields	33
3.1.1 Polar and directional blow-up	33
3.1.2 Successive Blow-ups	35
3.1.3 Quasi-homogeneous blow-up and the Newton diagram	39

3.2	The finiteness result for analytic vector fields on S^2	41
3.3	The Dulac problem	
4	Bifurcations of Regular Limit Periodic Sets	
4.1	The return map	51
4.1.1	Return map for a periodic orbit	51
4.1.2	Return map near an elliptic point	52
4.2	Regular limit periodic sets of finite codimension	55
4.2.1	Periodic orbit	55
4.2.2	Elliptic focus	57
4.3	Regular limit periodic set of infinite codimension	59
4.3.1	The Bautin Ideal	60
4.3.2	Properties of the Bautin Ideal	63
4.3.3	Finite cyclicity of regular limit periodic sets	67
4.3.4	Melnikov functions	72
4.3.5	Application to quadratic vector fields	75
4.3.6	Some properties of Abelian Integrals	82
5	Bifurcations of Elementary Graphics	
5.1	Transition map near a hyperbolic saddle point	92
5.1.1	Normal form of X_λ near the saddle point	92
5.1.2	The structure of the transition map for the normal family	94
5.1.3	The structure of the transition map of X_λ	104
5.2	Unfoldings of saddle connections in the finite codimension case . .	109
5.2.1	The codimension 1 case	109
5.2.2	The k -codimension case, $k \geq 2$	112
5.2.3	Bifurcation diagrams for generic unfoldings	117
5.3	Unfoldings of saddle connections of infinite codimension	125
5.3.1	Finite cyclicity property of analytic unfoldings	125
5.3.2	An example in quadratic vector fields	136
5.4	Unfoldings of elementary graphics	139
5.4.1	Hyperbolic graphic with 2 vertices	139
5.4.2	Generic unfoldings of hyperbolic k -graphics	142
5.4.3	Generic elementary polycycles	146
6	Desingularization Theory and Bifurcation of Non-elementary Limit Periodic Sets	
6.1	The use of rescaling formulas	151
6.2	Desingularization of unfoldings of cuspidal loops	155

6.2.1	Global blowing-up of the cusp unfolding	158
6.2.2	Asymptotic form for the shift map equation	162
6.2.3	Properties of $g_\varphi(\Omega)$	169
6.2.4	Monotonicity property for $t_0(x)$ and $t_\pi(x)$	171
6.2.5	Indications for the proof of Theorem 31	172
6.3	A method of desingularization for analytic vector fields	181
6.3.1	Foliated local vector fields	181
6.3.2	Operations of desingularization	184
6.3.3	Conjectures	186
6.3.4	Final comments and perspectives	188
Bibliography		193
Index		201

Preface

Let $X(x, y) = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}$ be a differentiable planar vector field. By integrating the differential equation $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, one obtains *the flow* $\varphi(t; (x, y))$ of X (the trajectories with initial conditions $(x, y) \in \mathbb{R}^2$).

In spite of the restriction to dimension 2, flows of planar vector fields model many interesting phenomena, for example the Van der Pol equation: $\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0$, which models an electric circuit including a triode valve; or the Duffin equation: $\ddot{x} + k\dot{x} - x + x^3 = 0$, which models a buckled beam (see [GH]). These second order differential equations are equivalent to differential equations associated to vector fields defined on $\mathbb{S}^1 \times \mathbb{R}$. This last set can be identified with an open subset of the plane. Of course, these equations do not include any exterior forcing. If we add a forcing periodic term to the equation (for instance $\varepsilon \cos \Omega t$), the system would be equivalent to an equation of a vector field in $\mathbb{S}^1 \times \mathbb{R}^2$, i.e., in a three-dimensional space, with a much more complicated dynamic than in the plane (it may have some chaotic orbits). Nevertheless, even in this case the study of the unforced equation will be an unavoidable first step in the study of the weakly forced equation.

The flow of a planar vector field X is a very simple example of a dynamical system. In fact, by the Poincaré-Bendixson theorem, every recurrent orbit of the flow φ is trivial, i.e., a singular point or a periodic orbit. Hence, it is quite easy to describe *the phase portrait* of the vector field X once one knows these trivial orbits. Recall that the phase portrait of X is the equivalence class of X up to the action of homeomorphisms sending oriented orbits to oriented orbits.

Let us consider for instance, the equation of a simple pendulum with a linear dissipation,

$$\ddot{x} + k\dot{x} + \omega^2 \sin x = 0, \text{ with } k > 0.$$

In the phase space $\mathbb{S}^1 \times \mathbb{R}$ with coordinates $x \in \mathbb{S}^1$ and $y = \dot{x} \in \mathbb{R}$, this equation is equivalent to the equation of the vector field,

$$X = y \frac{\partial}{\partial x} - (ky + \omega^2 \sin x) \frac{\partial}{\partial y}.$$

As a consequence of the Poincaré-Bendixson theorem, the phase portrait of X is uniquely determined by the following observations. There exist only two singular points: a stable focus at $(0, 0)$ and a saddle point at $(\pi, 0)$. Next, there exist no periodic orbits, because the divergence of X is equal to k . Finally, the ω -limit set of any orbit must be a singular point. The phase portrait determined by these observations is represented in Figure 1.

Figure 0.1

Hence, in order to know the dynamics of the flow, it is crucial to determine the position and the nature of its *singular elements* (the singular points and the periodic orbits). The difficulty of this task depends heavily on the way the vector field is defined. For instance, if the vector field is polynomial, i.e., if the components P, Q of the vector field are polynomials, the determination of the singular points is an algebraic question. Moreover, one can obtain the phase portrait of X near each singular point, at least when this point is algebraically isolated, using Seidenberg's theorem which can be translated into an algebraic algorithm.

On the other hand, even in the simple case of polynomial vector fields, it is, in general, impossible to determine explicitly, i.e., with an algebraic algorithm, the position and the nature of periodic orbits. Roughly speaking, the reason is the following: periodic orbits are obtained by equations in terms of flow, which cannot in general be computed explicitly. As a consequence, even if the dynamic generated by the flow of a planar vector field is very simple, its explicit determination is not possible in general. To study it, one has to use the qualitative ideas first introduced by Poincaré [Poi]. In some cases, they enable a study of the flow, without having to integrate the differential equation.

In this text, we study *families of planar vector fields* (X_λ) , depending on a parameter $\lambda \in \mathbb{R}^k$. The phase space will be $\mathbb{R}^2$ or, more generally, a surface of genus 0. Parameters enter naturally into the description of the system which is modeled by a differential equation (the temperature of ambient space, the length

of the pendulum which determines the oscillation ω in the above example, etc. ...). However, even if one is only interested in the study of vector fields, it may be interesting to consider vector fields which depend on parameters. For instance, it will be convenient to consider the polynomial vector fields of degree $\leq n$ as a family X_λ^n , whose parameter λ belongs to the set of coefficients of the two components P, Q .

Let us consider a family (X_λ) . For each value of λ , one can consider the phase portrait of X_λ . As we have suggested above, if X_λ has only a finite number of singular elements, the phase portrait of X_λ on each compact region has a combinatorial description. However, in order to understand the family, it is not sufficient to understand the phase portrait of each X_λ . This would be useless. We need the less static approach of *bifurcation theory*. This is in the study of how the phase portrait changes (bifurcates) when the parameter λ varies in the parameter space. Let us give a rough idea of this problem. For a given family (X_λ) , there is a closed subset $\Sigma \subset \mathbb{R}^k$, the parameter space, for which the phase portrait of X_λ changes. In general, Σ is a stratified set and the degeneracy of X_λ increases with the codimension of the strata to which λ belongs. The open set $U = \mathbb{R}^k \setminus \Sigma$ is the set where X_λ is structurally stable (this means that the phase portrait of X_λ does not change for small perturbations of λ). If $\lambda \in U$, and restricting ourselves to a compact domain in the phase space, X_λ has only a finite number of singular elements: singular points and isolated periodic orbits, which are also called *limit cycles*. When λ crosses Σ one may have bifurcations which imply a change in the number of some singular elements.

The difference that we have pointed out between the study of singular points and the study of periodic orbits for an individual vector field is even more important in the study of their bifurcations in families: if we consider a polynomial family, the bifurcations among singular points are again given by algebraic algorithms and this will not be the case, in general, for the limit cycles.

In order to study bifurcations of limit cycles we have first to detect where in the phase space such bifurcations can occur. Of course, it is there where limit cycles accumulate. Hence, it is natural to introduce these sets of accumulation. We call them *limit periodic sets* of the family. More precisely, a limit periodic set Γ , for the value λ_* of the parameter, will be any limit of a sequence of limit cycles $(\gamma_n)_n$. The limit of this sequence is taken in the Hausdorff topology of the set of all compact subsets of $\mathbb{R}^2$. We suppose that each γ_n is a limit cycle for a vector field X_{λ_n} , and that the sequence $(\lambda_n)_n$ tends to some λ_* in the parameter space. To understand the behaviour of the limit cycles in the family (X_λ) , one has to understand how they bifurcate from each periodic limit set Γ of the family.

The study of bifurcations of limit periodic sets in families of planar vector fields is precisely the subject of this book.

In this bifurcation problem, the first step is to obtain a bound for the number of limit cycles which bifurcate from Γ . Such a bound is called the *cyclicity* of Γ

in (X_λ) and is denoted by $Cycl(X_\lambda, \Gamma)$. Of course, this bound depends only on the germ of the family along Γ , also called the *unfolding* (X_λ, Γ) . It is easy to find differentiable unfoldings of limit periodic sets with infinite cyclicity. Hence, in order to be able to prove that the cyclicity is finite, we have to restrict ourselves to typical unfoldings. A general conjecture is that the cyclicity is finite for any *analytic* unfolding. It will be proved in the text that this conjecture would imply a positive answer to the “existential” Hilbert’s 16th problem:

“Prove that for any $n \geq 2$, there exists a finite number $H(n)$ such that any polynomial vector field of degree $\leq n$ has less than $H(n)$ limit cycles”.

Besides general questions, as is the above conjecture or Hilbert’s problem, it is important to be able to compute the cyclicity for explicit typical unfoldings (X_λ, Γ) . A famous example of such an explicit computation is the result of Bautin [B], proving that each focus or center singular point in the quadratic polynomial family X_λ^2 has a cyclicity of at most three.

Once one has detected the periodic limit sets of a family and computed their cyclicity, it remains to study their local bifurcations in the family. In general, it is then easy to finish the study of the *bifurcation diagram*, i.e., to obtain the set Σ and to describe the bifurcations occurring by crossing this set. To illustrate these general considerations, we want to introduce now a simple example of a family: the *Bogdanov-Takens unfolding*. This example only depends on two parameters and the diagram of bifurcation is easy to describe, but the study is sufficiently nontrivial to give a good idea of the general problem. Moreover, this example will be used throughout the text to illustrate the different methods we want to introduce.

The Bogdanov-Takens unfolding is a local 2-parameter family near a singular point of a vector field X_0 whose 2-jet is equivalent to:

$$X_0^{N\pm} \begin{cases} \dot{x} = y \\ \dot{y} = x^2 \pm xy. \end{cases}$$

The linear part of this vector field at $0 \in \mathbb{R}^2$ is nilpotent. One says that this singular point has *codimension 2*, because nilpotent, (2×2) -matrices form a codimension 2 subset in the space of all (2×2) -matrices. As a consequence, the germ of X_0 cannot be destroyed by a small perturbation when it appears in 2-parameter families of vector fields.

The following 2-parameter quadratic family unfolds $X_0^{N\pm}$:

$$X_\lambda^{N\pm} \begin{cases} \dot{x} = y \\ \dot{y} = x^2 + \mu + y(\mu \pm x) \end{cases} \quad \text{where } \lambda = (\mu, \nu) \in \mathbb{R}^2.$$

We will prove in the text that this unfolding of X_0 is *versal* in the sense that any other unfolding of X_0 factorizes through $X_\lambda^{N\pm}$ (up to some equivalence

relation). As the two unfoldings $X_\lambda^{N\pm}$ are equivalent up to the orientation of trajectories and a linear change of parameter, we will only consider the unfolding X_λ^{N+} . It will simply be written X_λ^N .

Here we only want to describe the bifurcations of X_λ^N , and at the same time we give some heuristic idea of why all possible bifurcations of X_0 appear in the unfolding X_λ^N .

First of all, let us note that the phase portrait of X_0 near the origin is the same as the phase portrait of X_0^N , and that it represents an equilibrium point on a curve with a cusp shape passing through the origin. This is the reason why X_0 is often called the *Cusp singularity*. In fact, this phase portrait is the same as the phase portrait of the Hamiltonian vector field with Hamiltonian function $H_0 = \frac{1}{2}y^2 - \frac{1}{3}x^3$. Moreover, X_λ^N has the same singular points as the Hamiltonian unfolding $H_\mu = \frac{1}{2}y^2 - \mu x - \frac{1}{3}x^3$. For this Hamiltonian function, we obtain a pair of singular points (a center and a saddle point) where μ becomes negative. For the vector field belonging to the family, the center is replaced by a focus or a node (see Figure 2). The change of the center to a focus or a node can be explained by the effect of the second parameter ν which modifies the divergence of X_λ .

Figure 0.2

We now want to search for the possible limit cycles of X_λ and their bifurcations along the limit periodic sets of the family. Note that any limit cycle must surround the focus point C_λ and leave the saddle point s_λ outside the disk it bounds. As a consequence we have only three possible limit periodic sets:

- the focus point e_λ ;
- a limit cycle γ_λ ;
- a homoclinic saddle connection Γ_λ at the saddle s_λ .

A rather profound result about the family X_λ is that the cyclicity of any limit periodic set is always equal to one. We will see that this result is obtained by different methods depending on the nature of the limit periodic set: an algebraic computation proves that a Hopf bifurcation can occur at the focus point; the limit cycles are studied using the properties of abelian integrals, and the cyclicity of the saddle connection is obtained using asymptotic properties of a non-differential return map. As a consequence of this cyclicity property it is easy to see that for any $\mu < 0$, a unique unstable limit cycle is created by a *Hopf bifurcation* when ν crosses some value $\nu_H(\mu)$ in a decreasing direction. This limit cycle grows bigger as ν decreases from this value $\nu_H(\mu)$, and finally reaches the saddle point and creates a *saddle connection* for some value $\nu_C(\mu) < \nu_H(\mu)$. No limit cycles exist for ν outside the interval $[\nu_C(\mu), \nu_H(\mu)]$.

These considerations permit us to draw the bifurcation diagram of Figure 0.3. We observe that there are three lines of bifurcations: the axis $\{\mu = 0\}$, which is a line of saddle-node bifurcations *SN* along which a pair of singular points bifurcates, and two curves H and C (graphs of the functions $\mu \rightarrow \nu_H(\mu)$, $\nu_C(\mu)$ respectively, with $\mu < 0$), branching at the origin of the parameter space. These curves have a quadratic contact with *SN*. There exists one and only one limit cycle in the tongue between the two curves H , C . It is clear from the above description of the behaviour of the limit cycles in a function of the variation of the parameter, that the curve H is a line of Hopf bifurcations and C is a line of saddle connections. We can summarize this bifurcation diagram by saying that it explains how a small limit cycle is created near the equilibrium point and dies at the saddle connection, when the parameter varies. It is also interesting to note that this limit cycle is unstable and is the boundary of a small basin of attraction of a stable focus point. This small basin of stability disappears at the Hopf bifurcation, but is transformed into a thin region between separatrices of the saddle point, when the limit cycle arrives to the saddle connection (see Figure 0.3).

The study of bifurcations in any unfolding X_λ follows the same general lines. First one has to look for a possible model given by a polynomial family X_λ^N . This model family ($X_\lambda^{N\pm}$ in our example) is obtained, in general, using the *theory of normal forms* (see for instance [CLW], [D2], for general information about normal forms) and a suitable truncation of this normal form to a *sufficient finite jet*. Next, we study the bifurcation of singular points in our model family X_λ^N . As mentioned above, as X_λ^N is a polynomial family, this study can be made using algebraic algorithms. We return to the initial family X_λ by using implicit function theorem arguments. Finally, we study the limit cycles of the family, looking first for the possible limit periodic sets, and then computing their cyclicity and studying their local bifurcations. Sometimes, this reduces again to algebraic algorithms. This is the case for instance for the “Hopf-Takens” bifurcations of limit cycles at a weak focus. However, in general, the study of bifurcations of limit periodic sets is more involved. Several different methods are needed from analytic geometry, the theory of abelian integrals, the theory of asymptotics of some special classes of non-

Figure 0.3

differentiable function families, and desingularization theory. *The aim of this text is to present these methods in a systematic way:* first, the analytic geometry used for *regular limit periodic sets*, next, the asymptotic methods which are related to the study of *elementary limit periodic sets*, and finally, the desingularization method applied to reduce *non-elementary limit periodic sets* to elementary ones.

We will not present a systematic review of possible generic bifurcation diagrams in functions of the number of parameters. Such a presentation may be found easily in the literature (see for instance [A], [GH], [CLW], [D], [DRS1], [DRS2], [KS]). The bifurcations which are presented in the text are given just to illustrate the methods. This is the case for the Bogdanov-Takens bifurcation which will be encountered several times, and for the bifurcations of the cuspidal loop, which is presented in the last chapter to illustrate the desingularization method.

An important application of the text is Hilbert's sixteenth problem. We have already mentioned that this problem can be replaced by a more general conjecture about the finite cyclicity of analytic unfoldings of limit periodic sets. This idea

motivates the study of all possible limit periodic sets in polynomial families and then the study the cyclicity of these “polynomial” limit periodic sets. In Chapter 2, we give the results of this scheme for the special case of quadratic vector fields.

The plan of the text is as follows.

In Chapter 1, we recall some general properties of vector fields on surfaces of genus 0, such as the Poincaré-Bendixson theorem, and we make a first approach to their bifurcation theory.

Limit periodic sets, which are the main subject of the text, are introduced in Chapter 2, where their general properties are established. We prove a first and partial result about their structure, in the spirit of the Poincaré-Bendixson theory, and define the cyclicity $Cycl(X_\lambda, \Gamma)$. We show that Hilbert’s problem reduces to a general conjecture on the finite cyclicity for analytic unfoldings. We describe this reduction in detail for quadratic vector fields.

In Chapter 3, we consider a single vector field (the “0-parameter case”) and review the desingularization theory and the solution of the Dulac problem asking if an analytic vector field on S^2 has only a finite number of limit cycles. The reason for this survey is that some ideas that we introduce in Chapter 3 will be extended to families in the next chapters. For the Dulac problem, we restrict ourself to the case of hyperbolic graphics which can be easily studied and generalized to families.

The last three chapters are devoted to the study of bifurcations of limit periodic sets and to partial proofs of the conjecture on finite cyclicity. In Chapter 4, we consider regular limit periodic sets, i.e., elliptic singular points and periodic orbits. They can be studied by standard methods of differential and analytic geometry. Of particular interest is the notion of the Bautin ideal associated with analytic unfoldings of ∞ -codimension vector fields of “centre type”. This ideal, first introduced by Bautin for the study of polynomial quadratic vector fields, is extended here to general analytic unfoldings. Using it, one can obtain an estimate for the cyclicity. We also relate it to Melnikov functions and give some applications to quadratic vector fields.

In Chapter 5, we consider elementary graphics which are the limit periodic sets whose singular points are elementary. We prove a general structure result for the unfoldings of the transition map near an hyperbolic saddle. This result, together with the notion of the Bautin ideal, allows a complete study of the saddle connection unfoldings. In the last part of the chapter, we present a review of some recent results by Mourtada, El Morsalani, Il’Yashenko, Yakovenko and others about general elementary graphics.

The last chapter is devoted to non-elementary limit periodic sets and their desingularization. We treat the simplest case, a connection at a Bogdanov-Takens cusp point, in detail and give some indication about a general desingularization theory for families of vector fields.

The principal aim of this text is to bring together results from previous articles, although some results appear here for the first time. For instance, Theorems 4.2 and 5.13 give explicit bounds for the cyclicity in terms of the Bautin ideal

and are applied in theorem 5.14 to quadratic vector fields. Section 6.2 presents results on generic bifurcations of the cuspidal loop. More detailed results about the cuspidal loop will appear in a forthcoming publication.

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